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Impatience and equilibrium indeterminacy

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Abstract

This paper extends the neoclassical one-sector optimal growth model by postulating that individual agents' time preference (impatience) depends on the economy-wide average consumption and average income, which are social factors taken as external by individual agents. The paper shows that with the socially determined individual time preference local equilibrium indeterminacy can arise. Moreover, local indeterminacy can also be associated with global indeterminacy in many cases. The results hold in models with bounded and unbounded growth.
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1. Introduction

This paper explores the implications of the hypothesis that individual agents' time preference is largely determined by the surrounding environment – that is, by social forces that are viewed as entirely external and cannot be controlled by individual agents. We consider an extension of the one-sector neoclassical growth model with optimizing infinitely lived agents, by specifying a dependence of the individual discount rate on the social variables: the economy-wide average consumption and

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average income.[1] As these aggregate variables are taken as exogenously given by the agent, the discount rate and the time-preference rate are equal at any moment just as in the case of a constant discount rate, but they both vary over time.[2]

The influence of social forces on individual time preference was emphasized by earlier classical economists. Notably, Rae (1834), using an alternative expression 'the effective desire of accumulation' for time preference, saw culture as a critical determinant of differences in time preferences across various economies: 'The mass of the individuals composing any society, being operated on the same causes, and having similar manners, habits, and to a great extent feelings also, must approximate to each other, in the strength of their effective desires of accumulation' (1834, p. 198).[3] Specifying the dependence of individual time preference on relevant aggregate variables represents a simple and tractable approach of modeling social and cultural influence. It stands in contrast to earlier models with endogenous time preference, such as Uzawa (1968) and Epstein (1987), where the discount rate is assumed to depend on variables taken as internal by the agent.

The paper is focused on the implications of the hypothesis of the socially determined time preference on the equilibrium dynamics. Within the extended neoclassical framework we find that the perfect-foresight equilibrium may not be unique, and that local indeterminacy can arise. Moreover, local indeterminacy can also be associated with global indeterminacy in many cases. Thus, with the socially determined time-preference optimality conditions are not sufficient to pin down the equilibria. This therefore describes a new class of models with indeterminacy. The results hold in models with bounded and unbounded growth.[4]

We consider here the influence of average consumption and average income on individual time preference because the connections between individual time preference and *individual* consumption or *individual* income have been extensively discussed in the literature. As noted above, Uzawa (1968) postulates that an agent's discount rate depends on the level of individual consumption, and such a specification has been widely adopted and extended in subsequent research (e.g., Lucas and Stokey, 1984; Epstein, 1987; Obstfeld, 1990). On the other hand, since Fisher (1930) it has long been argued both theoretically and empirically that an agent's rate of time preference is closely related to his income or wealth level, and a consensus that seems to emerge is that the rich are more patient. For example, Hausman (1979), Lawrance (1991) and Samwick (1998) all find from empirical

[1]Or the aggregate values of these variables with a population of unit mass. We can thus use the words 'average' and 'aggregate' interchangeably.

[2]See the next section for more formal arguments of why the two rates are equal.

[3]Fisher (1930) also mentions the connection between culture and time preference.

[4]Local indeterminacy implies the existence of stationary sunspot equilibria. For the case of unbounded growth, an implication of the indeterminacy result is that countries with similar fundamentals and initial conditions can nevertheless save and grow at different rates. See, Benhabib and Farmer (1999) for an extensive survey for models with indeterminacy and sunspots.

evidence that rates of time preference vary inversely with households' income. Becker and Mulligan (1997) present a theoretical model which predicts that the wealthy are more patient.[5] We therefore consider the polar case by incorporating the two aggregate variables, while leaving out altogether variables taken as internal by the agent. An implication of this formulation is that an agent's time preference conforms to that of the society, and that no individual decision can have an appreciable effect on these external forces. This specification captures an important feature of actual individual behavior as people tend to be largely influenced by social norms.

This paper is not the only nor the first attempt to model individual time preference that is socially determined. Shi (1999) postulates that the individual discount rate relies on average consumption habit, and Schmitt-Grohé and Uribe (2003, pp. 169–170) place average felicity in the discount-rate function in a model with endogenous labor supply. Ogawa (1993) estimates a model where the discount rate is assumed to depend on average labor income.[6] Drugeon (1998) considers a model with the discount rate depending on individual and average consumption, while Drugeon (1996a, Section IV) and Palivos et al. (1997, pp. 220–221) study models by specifying dependences of individual discount rates on individual consumption and average capital holdings.

The structure of the paper is as follows. In Section 2 we consider an extension of the Ramsey–Cass–Koopmans model by assuming that the representative agent's discount rate depends on average consumption and average income. In this setup we prove our main result: local indeterminacy can arise if the individual rate of time preference increases with average consumption and decreases with average income. Moreover, with a strictly concave production function multiple steady states can occur. In Section 3 we present two variations of the model in Section 2. We first study a setup in Section 3.1 in which the discount rate depends solely on average consumption and the production is subject to externalities from average capital stock, and in Section 3.2 we consider the case with endogenous labor supply assuming that the discount rate relies only on average income. In both extensions we show that local as well as global indeterminacy can happen. In Section 4 we study a model with endogenous growth, and demonstrate that an indeterminate balanced growth path (and multiple balanced growth paths in some cases) can arise when the discount-rate function is homogenous of degree zero in average consumption and average income. Section Appendix A offers the concluding remarks.

[5]Becker and Mulligan (1997, Section V) also contains an extensive overview of the empirical studies on the connections between wealth and time preference.

[6]While recognizing its analytical simplicity, neither Schmitt-Grohé and Uribe nor Ogawa elaborates on the economic implications of the assumption that the rate of individual time preference depends on aggregate or average variables. In a broad context, models with socially determined time preference are related to the recent literature known as ''social economics' (e.g., Becker and Murphy, 2000; Durlauf and Young, 2001), which emphasizes the role of social influences on individual preferences and cognition. Here, it is the discount rate rather than the felicity that is influenced by social variables.

2. A model with socially determined time preference

2.1. The model

We consider an extension of the one-sector neoclassical growth model with optimizing infinitely lived agents. The economy is made up of a large number of identical agents, normalized to unity, and each of them seeks to maximize the present discounted value of the lifetime utility

$$U_0 = \int_0^\infty u(c_t)\exp\left[-\int_0^t \rho(C_v, Y_v)\,\mathrm{d}v\right]\mathrm{d}t, \tag{1}$$

where c_t is the representative agent's consumption at time t, C_v denotes the economy-wide average level of consumption (consumption standard) at time v. Y_v denotes the economy-wide average level of current income (living standard) at time v, to be made more precise below. $u(\cdot)$ is the felicity, and $\rho(\cdot,\cdot)$ is the subjective discount-rate function (or 'discounting function' for short).

Assumption 2.1. (i) $u(c)$ is strictly positive and strictly concave, that is, $u>0$, $u'>0$, $u''<0$[7];
(ii) $\rho>0$ and ρ is linear in (C, Y).

Although the discount rate is nonconstant, it depends on the social variables: average consumption and average income, which are taken as exogenously given by the representative agent. Thus, the agent accepts passively the social norms in his valuation of future happiness relative to today's welfare. The linearity assumption for ρ in (ii) of Assumption 2.1 is adopted for the purposes of simplification and specificity, which is rich enough for obtaining the main results below.[8]

An immediate implication of the above specification for the discounting function is that the agent's subjective discount rate and time-preference rate are equal at any point in time. To see this, following Epstein (1987), define the (local) rate of time preference at time s as

$$\varsigma(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log U_s|_{\dot{c}(s)=0}. \tag{2}$$

[7]Technically the major results below do not require the assumption of $u(c)>0$. It is however sometimes convenient for expositional purposes. In addition, if $u(c)$ has a lower bound, we can always add a sufficiently large constant, say, Σ, to make $(u+\Sigma)$ to be uniformly positive. For the commonly used CES felicity, $(1-\sigma)^{-1}c^{1-\sigma}$, if $\sigma>1$ it is always negative and has no lower bound (for $\sigma=1$, i.e., the utility function that is logarithmic in consumption, there exists no lower bound either). Thus, when we consider examples with a CES felicity in this paper, we limit to the case of $0<\sigma<1$. The assumption of a positive felicity is adopted in Drugeon (1996a, 1998) and Shi (1999). See also footnote 15 for the case with endogenous leisure.

[8]With linearity assumption, ρ can be negative for negative coefficient(s) for C or/and Y. Our assumption $\rho>0$, however, is a local concept, that is, it holds at and near a steady state.

U_s is the marginal utility of consumption near time s, computed via the Volterra derivative:

$$U_s = u'(c(s)) \exp\left[-\int_0^s \rho(C_v, Y_v)\,\mathrm{d}v\right].$$

Calculation reveals that

$$\varsigma(s) = \rho(C_s, Y_s), \tag{3}$$

where the RHS is the discount rate at time s. This property of our model is in stark contrast to those of previous models with the discounting functions depending on variables internal to individual agents where, unless at a steady state, the two rates generally do not coincide and the rate of time preference depends on an additional costate variable (equal to the maximized lifetime utility functional). As will be clear below, in the absence of the influence of the extra costate, characterizing the equilibrium is considerably simpler here than in the Uzawa–Epstein type models with endogenous time preference.

The agent's capital at time t is denoted by k_t; he also has one unit of labor which is supplied inelastically. The rental rate for capital and wage rate are r_t and w_t, respectively. For simplicity, assume that there is no physical depreciation of capital. The agent, given k_0, chooses paths for consumption, c_t, to maximize (1), subject to the budget constraint

$$\dot{k}_t = r_t k_t + w_t - c_t. \tag{4}$$

The single good is assumed to be produced with a standard neoclassical technology $F(k, l)$ that uses capital k and labor l as inputs, i.e.,

Assumption 2.2. The production function $F(k, l) > 0$ exhibits constant returns to scale in k and l and is concave.

Denoting the production function's intensive form as $f(k) \equiv F(k, 1)$, from Assumption 2.2 we have[9]

$$f'(k_t) > 0, \quad f''(k_t) \leqslant 0. \tag{5}$$

Perfect competition in factor markets implies that $r_t = f'(k_t)$; $w_t = f'(k_t) - k_t f'(k_t)$. The representative agent's income at time t, $y_t = f(k_t)$. Denoting K_t as the average capital holdings, the economy-wide average income at time t is[10]

$$Y_t = f(K_t). \tag{6}$$

Setting up the present-value Hamiltonian of the problem, $\mathscr{H} = u(c_t) \exp[-\int_0^t \rho(C_v, Y_v)\,\mathrm{d}v] + \mu_t(r_t k_t + w_t - c_t)$, first-order conditions with respect to consumption c_t and the costate variable μ_t are (in equilibrium $C_v = c_v$, $Y_v = y_v$):

[9]Assumption 2.2 nests the linear technology case (i.e., $f''(k) = 0$). The linear technology case can be alternatively interpreted as one of a small open economy where the agent has perfect access to international goods and capital markets, and he can borrow and lend at a constant interest rate.

[10]Note that since Y is a function of K, we could alternatively assume in (1) that the discount rate is a function of average consumption and average capital holdings, i.e., $\rho = \rho(C, K)$.

$u'(c_t)\exp(\int_0^t -\rho(c_v, y_v)\,\mathrm{d}v) = \mu_t$, and $\dot{\mu}_t = -\mu_t r_t$, which, by introducing the current-value costate variable, $\lambda_t = \mu_t \exp(\int_0^t \rho(c_v, y_v)\,\mathrm{d}v)$, can be rewritten as

$$u'(c_t) = \lambda_t, \tag{7}$$

$$\frac{\dot{\lambda}_t}{\lambda_t} = \rho(c_t, f(k_t)) - f'(k_t). \tag{8}$$

The system can be reduced to two dynamic equations,

$$\dot{c}_t = \frac{u'(c_t)}{u''(c_t)}[\rho(c_t, f(k_t)) - f'(k_t)], \tag{9}$$

$$\dot{k}_t = f(k_t) - c_t, \tag{10}$$

together with the transversality condition. Eq. (9) is the intertemporal Euler equation, and Eq. (10) is the resource constraint. Note that compared with the standard Ramsey–Cass–Koopmans model with a constant discount rate, the only difference is that here in Eq. (9) the discount rate depends on consumption and capital, whereas the resource constraint equation (10) remains unchanged. Compared with models of endogenous impatience with the discounting functions depending on individual variables, the dynamic system here contains one less costate variable and hence has one less dimension.

2.2. Indeterminacy conditions

We note first from Eqs. (9) and (10) that in this growth model with endogenous time preference there may exist one or multiple (or even no) steady states. Below we assume that there exists at least one steady state, and denote the steady state(s) generally as (c^*, k^*). Linearizing Eqs. (9) and (10) we obtain the following linear system:

$$\begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \begin{bmatrix} \frac{u'}{u''}\rho_C & \frac{u'}{u''}(\rho_Y f' - f'') \\ -1 & f' \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}. \tag{11}$$

The trace and determinant of the Jacobian matrix of this linear system are given by the expressions:

$$\mathrm{tr} = \frac{u'}{u''}\rho_C + f', \quad \det = \frac{u'}{u''}(\rho_C f' + \rho_Y f' - f''). \tag{12}$$

Since the system (11) contains only one predetermined variable, k_t, the perfect-foresight equilibrium is locally indeterminate if both eigenvalues of the matrix have negative real parts, or equivalently, if the trace is negative and the determinant is positive. We thus have the following result:

Proposition 2.1. *If, evaluated at a steady state,*

$$\left(\frac{u'}{u''}\rho_C + f'\right) < 0, \tag{13}$$

$$(\rho_C f' + \rho_Y f' - f'') < 0, \tag{14}$$

then the equilibria near the steady state are indeterminate.

In Proposition 2.1, under conditions (13) and (14), first-order and transversality conditions are not sufficient to determine a unique solution path. Indeterminacy of this sort implies that the rational expectations equilibria involve random variables, unrelated to the economy's fundamentals, simply because agents believe it to be so.

From (13) and (14), the following two conditions are necessary for the indeterminacy result:

$$\rho_Y < 0, \tag{15}$$

$$\rho_C > 0. \tag{16}$$

Condition (15) implies that the discount rate or the rate of time preference (as the two rates are equal) falls with the economy's level of living standard, that is, as the economy gets wealthier, the agent becomes more patient and is more willing to defer consumption. Condition (16), on the other hand, says that the rate of time preference increases with average consumption, that is, as the society consumes more, the agent himself becomes even more impatient and is less willing to defer consumption. Note that under the assumption $u > 0$, high level of living standard raises the agent's lifetime utility, whereas high average consumption reduces his lifetime utility. Thus the two conditions imply that the society's living standard is like a public good that yields a positive externality to the agent's utility, and the society's consumption is like a public good that delivers a negative externality.[11]

In the standard Ramsey–Cass–Koopmans model with a constant discount rate and a strictly concave production function, we have $\rho_C = \rho_Y = 0$, and $f'' < 0$. In that case the determinant of the Jacobian matrix is negative and therefore the equilibrium is locally determinate. Generally, the equilibrium is locally unique if $(\rho_C f' + \rho_Y f' - f'') > 0$.[12]

To see more clearly how indeterminacy comes about, let us consider first a case in which local indeterminacy does *not* arise but the system is explosive. In particular, when $\rho_C = 0$, $\rho_Y < 0$ and $f'' = 0$, both the trace and the determinant are positive, and there exists no locally convergent equilibrium solution. Intuitively, if everyone's income (average income) is higher on any given date, the discount rate applied to future utility will be lower (i.e., the agent will value future utility more), so that the marginal utility of future consumption (and hence the marginal utility of future wealth) will be higher.[13] This will in turn induce further accumulation of capital

[11]As $\rho_C > 0$ implies that average consumption reduces the agent's lifetime utility, it represents one kind of specification of the so-called 'jealousy' effects. See, Shi (1999, p. 443) about this point. For a recent survey of the literature that models jealousy by incorporating average consumption in the felicity function, see, Dupor and Liu (2003).

[12]Two other special cases in which the equilibrium is locally unique are when the production technology is linear ($f'' = 0$) and either $\rho_C > 0$, $\rho_Y = 0$, or $\rho_C = 0$, $\rho_Y > 0$.

[13]We can illustrate the point using a discrete-time, two-period model, with the utility given by $U(c_0, c_1, \rho) = u(c_0) + 1/(1+\rho)u(c_1)$, where ρ is the discount rate. If ρ is decreasing in the average income of the first period, Y_0, then the cross partial $U_{Y_0 c_1} = (\partial/\partial Y_0)(\partial U/\partial c_1) = (\partial/\partial Y_0)[(1/(1+\rho))u'(c_1)] > 0$.

leading to even higher income. Thus, any equilibrium path starting away from the steady state does not locally converge.

Why in this example can indeterminacy happen if $\rho_Y < 0$ and at the same time $\rho_C > 0$? Consider starting from an equilibrium path. Suppose that, when the agent is optimistic that the permanent income will be higher (caused by $\rho_Y < 0$), the agent increases the initial consumption level by jumping onto a new equilibrium path.[14] As $\rho_C > 0$, if everyone's initial consumption level is higher, the discount rate applied to future utility will be higher so that the agent will value future utility less. Therefore, the marginal utility of future consumption will be lower, and if this factor dominates it will lead to a reversal of the over-accumulation of capital and income, and the new equilibrium path can indeed be a convergent one. Thus, indeterminacy requires two opposing forces at work: $\rho_C > 0$ acts as a stabilizing force, while $\rho_Y < 0$ is a destabilizing factor.

We next give several examples in which Proposition 2.1 holds. Let the linear discounting function be denoted as

$$\rho(C, Y) = \alpha C - \beta Y + \gamma \quad \text{where } \rho_C = \alpha \quad \text{and} \quad \rho_Y = -\beta. \tag{17}$$

Example 2.1. Begin by choosing the felicity and production technology as, $u(c) = 1 - e^{-\eta c}$, and $f(k) = Ak$. All the agent's income comes from capital rental and therefore $Y = AK$. From Eqs. (9) and (10), the following simple linear system can be obtained:

$$\dot{c}_t = \frac{1}{\eta}(-\alpha c_t + \beta A k_t + A - \gamma), \tag{18}$$

$$\dot{k}_t = Ak_t - c_t. \tag{19}$$

The steady state is $k^* = (\gamma - A)/A(\beta - \alpha)$, $c^* = (\gamma - A)/(\beta - \alpha)$. For the Jacobian matrix, the trace is $(A - (\alpha/\eta))$ and the determinant $(A/\eta)(\beta - \alpha)$. The trace is negative if (i) $(A - (\alpha/\eta)) < 0$, and the determinant positive if (ii) $\beta > \alpha$. Notice also that positive steady-state values for k and c require that (iii) $\gamma > A$. Thus indeterminacy happens if the parameters satisfy conditions (i)–(iii).

Example 2.2. Let $u(c) = (1 - \sigma)^{-1} c^{1-\sigma}$, where $0 < \sigma < 1$, and $f(k) = Ak$. The system reduces to two differential equations: $\dot{c}_t = c_t/\sigma(-\alpha c_t + \beta A k_t + A - \gamma)$ and $\dot{k}_t = Ak_t - c_t$. Unlike Example 2.1, the first equation is not a linear one, but the steady state remains the same (if other parameter values are the same and regardless the values of η and σ). For the Jacobian matrix of the linearized system, the trace is $A[1 - \alpha(\gamma - A)/\sigma A(\beta - \alpha)]$, and determinant $(Ac^*/\sigma)(\beta - \alpha)$. Therefore indeterminacy occurs if (i) $\alpha(\gamma - A)/\sigma A(\beta - \alpha) > 1$, (ii)$\beta > \alpha$. Note that from condition (i) the smaller the intertemporal elasticity in consumption σ is, the easier it is to have

(*footnote continued*)
Thus, a rise in Y_0 raises the marginal utility of period-2 (*future*) consumption. See Obstfeld (1990) for more on this point.

[14]See Wen (2001) for an intuitive understanding of the models with indeterminacy based on the permanent income approach. Additional intuitive interpretations related to this approach for the results in the present paper are provided in Section 3.2.

indeterminacy. This is so because indeterminacy arises when we consider constructing an alternative equilibrium we must initially change consumption, and the magnitude of σ is inversely related to the agent's desire to intertemporally smooth consumption. Similar explanations can also be given about the parameter η in Example 2.1.

Example 2.3. Let $u(c) = (1-\sigma)^{-1}c^{1-\sigma}$, $f(k) = Ak^{\theta}$, where $0<\sigma$, $\theta<1$. The agent's total income y, consisting of the interest rate and wage payment, is equal to Ak^{θ}. The average amount of income $Y = AK^{\theta}$. The system boils down to two differential equations, namely,

$$\dot{c}_t = \frac{c_t}{\sigma}(-\alpha c_t + \beta A k_t^{\theta} + \theta A k_t^{\theta-1} - \gamma), \tag{20}$$

$$\dot{k}_t = Ak_t^{\theta} - c_t. \tag{21}$$

The trace and the determinant of the Jacobian matrix of the linearized system are given by

$$\mathrm{tr} = A(k^*)^{\theta-1}\left(\theta - \frac{\alpha}{\sigma}k^*\right), \quad \det = \frac{A\theta c^*(k^*)^{\theta-2}}{\sigma}[(\beta-\alpha)k^* - (1-\theta)]. \tag{22}$$

Indeterminacy arises when $\beta>\alpha$ and for sufficiently large steady-state value, k^*, which can be solved from the equation: $\theta + (\beta-\alpha)k = (\gamma/A)k^{1-\theta}$. Normally there exist two steady states, of which the smaller one is a saddle point, whereas the larger one is indeterminate (but for some parameter values there may exist no or only one steady state). The result is intuitive. Although ρ is a decreasing function of k, for small values of k, in the Euler equation (9) the term f' dominates and equilibrium uniqueness obtains just as in the standard Ramsey–Cass–Koopmans model. For sufficiently large values of k, however, the term ρ dominates and local indeterminacy occurs. Thus in this case there are both local and global indeterminacy. A numerical example is: $\alpha = 0.012$, $\beta = 1$, $\sigma = 0.8$, $\theta = 0.3$, $\gamma = 0.6$, $A = 0.2$.

3. Two variations

In this section we present two variations of the model in Section 2. In particular, we present two models with the discount rate (or the time-preference rate) depending on either average consumption or average income alone. In Section 3.1, we assume the discount rate depends solely on average consumption while the production is subject to an externality from average capital stock. In Section 3.2, we consider a model with endogenous labor supply and with the discount rate relying only on average income, in the absence of production externalities as in Section 2. In both extensions we show that local as well as global indeterminacy can occur. In the course of the presentation in this section we also compare with some other one-sector growth models studied in the literature.

3.1. A model with technological externalities

As noted above, the indeterminacy result in Section 2 happens because the endogenous discount rate alters the behavior of the Euler equation, which in turn depends on the difference between the discount rate and the marginal product of capital. Indeterminacy occurs when the difference increases with consumption and decreases with capital. This suggests an alternative mechanism for indeterminacy. In particular, if production is subject to an externality from average capital stock K, and the discounting function depends only on average consumption C and increases with it, then the difference between the discount rate and the marginal product of capital can also increase with consumption and decrease with capital, making it possible for indeterminacy to occur.

Assumption 3.1. (i) The discounting function, $\rho = \rho(C) > 0$, and is linear in C; (ii) The production function $F(k, l, K)$, with respect to k and l, displays constant returns to scale and is concave. In addition, the social production function is an increasing function of k, i.e., $F_1(k, l, k) + F_3(k, l, k) > 0$.

Assumption 3.1(ii) nests the positive externality case (i.e., $F_3 > 0$); however, it also allows for negative externalities. Denoting the production function's intensive form as $f(k, K) \equiv F(k, 1, K)$, then $f_1(k, k) + f_2(k, k) > 0$. Solving the representative agent's problem gives

$$\dot{c}_t = \frac{u'(c_t)}{u''(c_t)}[\rho(c_t) - f_1(k_t, k_t)], \tag{23}$$

$$\dot{k}_t = f(k_t, k_t) - c_t. \tag{24}$$

By linearization we obtain

$$\begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \begin{bmatrix} \frac{u'}{u''}\rho' & -\frac{u'}{u''}(f_{11} + f_{12}) \\ -1 & f_1 + f_2 \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}. \tag{25}$$

The trace and determinant of the Jacobian matrix in (25) are

$$\mathrm{tr} = \frac{u'}{u''}\rho' + f_1 + f_2, \quad \det = \frac{u'}{u''}[\rho'(f_1 + f_2) - (f_{11} + f_{12})]. \tag{26}$$

Thus we have the following result:

Proposition 3.1. *If, evaluated at a steady state,*

$$\left(\frac{u'}{u''}\rho' + f_1 + f_2\right) < 0, \tag{27}$$

$$\rho'(f_1 + f_2) - (f_{11} + f_{12}) < 0, \tag{28}$$

then the equilibria near the steady state are indeterminate.

Since $(f_1 + f_2) > 0$, the following two conditions are necessary for indeterminacy:

$$\rho' > 0, \tag{29}$$

$$f_{11} + f_{12} > 0, \tag{30}$$

where condition (29) has the same economic implications as condition (16) in Section 2 – as the society's consumption increase the agent is less willing to defer consumption. Condition (30) says that the private marginal product $f_1(k,k)$ is increasing in k. The reason for the similarity between the indeterminacy result here and that in Section 2 can be seen from the Euler equations (9) and (23). The RHS of Eq. (9) depends on, $[\rho(c, f(k)) - f'(k)]$, whereas the RHS of Eq. (24) depends on $[\rho(c) - f_1(k,k)]$. In Section 2, indeterminacy relies on the discount rate being a decreasing function of k, so that $[\rho(c, f(k)) - f'(k)]$ is decreasing for some range of values of k. Here, condition (30) makes it possible that $[\rho(c) - f_1(k,k)]$ is decreasing in k. Thus, for indeterminacy to happen, while the stabilizing factor (i.e., the discount rate being an increasing function of average consumption) remains the same as that of the setup in Section 2, the increasing marginal product of capital replaces the condition $\rho_Y < 0$ in Section 2 as the destabilizing factor in the setup here. This destabilizing factor implies that the representative agent when expecting higher rate of return in capital would be willing to invest more, and the rate of return will indeed increase if all agents simultaneously increase their investment, inducing further incentive to save.

To see more clearly the results in this subsection and Section 2, we next compare the similarities and differences of various one-sector growth models, by directly looking at the corresponding Jacobian matrices for the linearized (c_t, k_t) systems.

3.1.1. Comparison of the Jacobian matrices of various one-sector models

A one-sector growth model can usually be reduced to two differential equations for consumption c_t and capital k_t, which characterize the dynamics of the model. Since indeterminacy requires the trace to be negative and the determinant positive, and the trace and the determinant are computed from the elements of the corresponding Jacobian matrix of a linearized system, one can directly examine the signs of the elements of the corresponding 2×2 matrix to see why indeterminacy can or cannot occur. Below we compare the Jacobian matrices of various one-sector models (with or without indeterminacy), along with the models in this subsection and Section 2. For each model assume that we have a linearized system for c_t and k_t, which can be generically represented by

$$\begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}, \tag{31}$$

where the first equation corresponds to the linearized Euler equation, and the second to the linearized resource constraint equation. We use as a benchmark case the standard Ramsey–Cass–Koopmans model with a constant discount rate and a strictly concave production function, in which the signs of elements of the Jacobian matrix are given by

$$\begin{bmatrix} 0 & - \\ - & + \end{bmatrix}, \tag{32}$$

where the element at the lower left is in fact '-1'. In this case, the determinant is negative (the trace is positive) and the steady state is a saddle point. We now look at in different models how the signs of the elements of the Jacobian matrices vary to affect the corresponding traces and determinants compared with this benchmark case.

(I) For the model in this subsection, under the necessary conditions for indeterminacy (29) and (30), the signs of the elements of the Jacobian matrix are

$$\begin{bmatrix} - & + \\ - & + \end{bmatrix}, \tag{33}$$

Compared with the benchmark case (32), in (33) j_{11} is negative and j_{12} becomes positive, while the signs for the elements in the second row remain unchanged. This makes it possible that the trace is negative and the determinant positive. Here, in the Euler equation, the negative sign for j_{11} (because $\rho' > 0$) acts as a stabilizing force whereas the positive sign for j_{12} (because $j_{11} + j_{12} > 0$) operates as a destabilizing factor.

(II) The Jacobian matrix for the linearized system in Section 2 has the same form as (33) in (I) under the necessary conditions (16) and (15). In that case $j_{11} < 0$ as $\rho_C > 0$, and $j_{12} > 0$ as $\rho_Y < 0$.

(III) Boldrin and Rustichini (1994) establish that, if the discount rate is constant, and for the one-sector models with exogenous labor supply and with a positive capital externality, local indeterminacy cannot arise regardless of the magnitude of the externality. In their case the Jacobian matrix is given by

$$\begin{bmatrix} 0 & * \\ - & + \end{bmatrix}, \tag{34}$$

whose trace is always positive because $\rho' = 0$, that is, the necessary condition (29) cannot satisfied. Note that only the sign for j_{12} can possibly change relative to (32).

(IV) Kehoe (1991) presents an example where indeterminacy happens for a model with a constant discount rate and when production is subject to a negative externality from capital (i.e., $f_2 < 0$). With the negative externality, it is possible that the private marginal product, $f_1(k, k)$, is increasing in k (i.e., $f_{11} + f_{12} > 0$) and the social production, $f(k, k)$, is decreasing in k (i.e., $f_1 + f_2 < 0$), so that the trace can be negative and determinant positive. The quadratic production function example given in Kehoe (1991, pp. 2133–2134) exactly satisfies these conditions. Under these conditions, the Jacobian matrix in Kehoe's model takes the form,

$$\begin{bmatrix} 0 & + \\ - & - \end{bmatrix}. \tag{35}$$

Compared with the benchmark case (32), j_{12} has a positive sign while j_{22} becomes negative. Thus the negative sign for (f_1+f_2) works as a stabilizing force in the resource constraint equation.

(V) Finally, Kehoe et al. (1992, pp. 60–61) give an example where indeterminacy happens when production is subject to an externality from average consumption, that is, the production function is $g(k,C)$. In that case the Jacobian matrix is $\begin{bmatrix} -\frac{u'}{u''}g_{12} & -\frac{u'}{u''}g_{11} \\ g_2-1 & g_1 \end{bmatrix}$, where $g_1>0$ and $g_{11}<0$. Indeterminacy is possible when $g_{12}<0$ and $(g_2-1)>0$, that is, the Jacobian matrix takes the form,

$$\begin{bmatrix} - & - \\ + & + \end{bmatrix}. \tag{36}$$

What is different from the benchmark case in (32) is that in (36) j_{11} is negative and j_{21} becomes positive. The condition $g_{12}<0$ as a stabilizing factor plays the same role as the condition that the discount rate is an increasing function of average consumption in the models in (I) and (II). On the other hand, the condition $(g_2-1)>0$ is a destabilizing force in the resource constraint equation. The example given by Kehoe, et al. satisfies these conditions.

We next give two examples in which Proposition 3.1 holds.

Example 3.1. Let the felicity, discounting and production functions be, $u(c)=1-e^{-\eta c}$, $\rho(C)=\alpha C+\gamma$, $f(k,K)=Ak^{\theta}K^{\chi}$, where η, A, $\chi>0$, $0<\theta<1$. Substituting these functions into (25), we obtain

$$\begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \begin{bmatrix} -\dfrac{\alpha}{\eta} & A(\theta+\chi-1)\beta(k^*)^{\theta+\chi-2} \\ -1 & A(\theta+\chi)(k^*)^{\theta+\chi-1} \end{bmatrix} \begin{bmatrix} c_t-c^* \\ k_t-k^* \end{bmatrix}. \tag{37}$$

The trace and determinant of the Jacobian matrix are

$$\text{trace} = \left[A(\theta+\chi)(k^*)^{\theta+\chi-1}-\frac{\alpha}{\eta}\right], \quad \det = \frac{A(k^*)^{\theta+\chi-2}}{\eta}[\theta(\theta+\chi-1)-\alpha(\theta+\chi)k^*]. \tag{38}$$

Indeterminacy requires that $\alpha>0$ and $(\theta+\chi)>1$. These are just the two necessary conditions (29) and (30). Normally, there are two steady states. Unlike Example 2.3, however, here the larger steady state is a saddle point, and the smaller one is indeterminate. This happens because indeterminacy requires the unstable factor, $A(\theta+\chi-1)\beta(k^*)^{\theta+\chi-2}$, be greater than the stable factor, $A\alpha(\theta+\chi)(k^*)^{\theta+\chi-1}/\eta$, and the former one dominates the latter for sufficiently small values of k^*. It is the other way around for the larger steady state. A numerical example is: $\alpha=0.015$, $\gamma=0.004$, $\theta=0.81$, $\chi=0.25$, $\eta=0.25$, $A=0.005$.

Example 3.2. Let $u(c)=(1-\sigma)^{-1}c^{1-\sigma}$, where $0<\sigma<1$, and all the other functions and assumptions remain the same as in Example 4. The trace and the determinant of the Jacobian matrix are, trace $=[A(\theta+\chi)(k^*)^{\theta+\chi-1}-(\alpha c^*/\sigma)]$, det $=Ac^*(k^*)^{\theta+\chi-2}/\sigma$ $[\theta(\theta+\chi-1)-\alpha(\theta+\chi)k^*]$. If all the other parameter values are the same (regardless of the values of η and σ), the steady-state solutions are identical with those in Example 3.1, and so are their properties. That is, the smaller steady state is indeterminate, whereas the larger one is a saddle point. Despite these similarities, indeterminacy requires very small values of σ or close to the linear felicity function. This is so because for small steady-state value of k^*, c^* is also small. Unlike Example 3.1, here c^* enters the expression for the trace, and in order for the trace to be negative σ must be sufficiently small. As a numerical example, indeterminacy occurs only when σ is close to 0.016 for the parameter values given in Example 3.1.

3.2. A model with endogenous labor supply

In the previous subsection, in the presence of technological externalities indeterminacy is possible when the discounting rate depends on average consumption alone. We now present another polar case. In particular, we consider a variation of the model in Section 2 to allow for endogenous labor choice, and show that indeterminacy can arise when the discounting rate depends solely on average income. We assume that there are no production externalities as in Section 2. The representative agent maximizes his lifetime utility, given by

$$\int_0^\infty u(c_t, l_t) \exp\left[-\int_0^t \rho(Y_v)\,\mathrm{d}v\right] \mathrm{d}t, \tag{39}$$

in which the felicity is

$$u(c_t, l_t) = (1-\sigma)^{-1} c_t^{1-\sigma} - l_t, \tag{40}$$

where we continue to assume that $0<\sigma<1$ as in the examples with inelastic labor supply above.[15] In addition, we make the following assumption:

Assumption 3.2. (i) $\rho > 0$ and ρ is linear in Y, denoted as $\rho(Y) = -\beta Y + \gamma$; (ii) The production function is Cobb – Douglas, i.e., $F(k,l) = Ak^{b_1} l^{b_2}$, where $A, b_1, b_2 > 0$, and $b_1 + b_2 = 1$.

The model here can be viewed as the general equilibrium version of the model estimated by Ogawa (1993), who assumes that the discount rate depends on average labor income. The results below show that in the general equilibrium framework multiple equilibria can arise in this type of models with preference externalities.

Solving the agent's maximization problem yields the following first-order conditions,

$$c_t^{-\sigma} = Ab_2 k_t^{b_1} l_t^{b_2-1}, \tag{41}$$

$$\dot{c}_t = \frac{c_t}{\sigma}(Ab_1 k_t^{b_1-1} l_t^{b_2} + \beta y_t - \gamma), \tag{42}$$

[15] Unlike the cases with exogenous labor supply, here even with $0<\sigma<1$, the utility given in (40) is still not uniformly positive. But we could add a sufficiently large constant Σ to make $u(c,l) + \Sigma$ to be positive at least in a range around the steady state(s). We should stress again, however, the assumption that $0<\sigma<1$ is not essential for the results below, and that indeterminacy can happen for $\sigma \geqslant 1$. See also footnote 7.

along with the resource constraint

$$\dot{k}_t = Ak_t^{b_1} l_t^{b_2} - c_t, \tag{43}$$

where the individual income $y_t = Ak_t^{b_1} l_t^{b_2}$. From Eq. (41), we have

$$l_t = (Ab_2)^{1/b_1} c_t^{-(\sigma/b_1)} k_t. \tag{44}$$

By substitutions, the system can be reduced to two dynamic equations for c_t and k_t:

$$\dot{c}_t = \frac{c_t}{\sigma}[b_1 A(Ab_2)^{\sigma b_2/b_1} c_t^{-(b_2/b_1)} + \beta A(Ab_2)^{\sigma b_2/b_1} c_t^{-(b_2/b_1)} k_t - \gamma], \tag{45}$$

$$\dot{k}_t = A(Ab_2)^{b_2/b_1} c_t^{-(\sigma b_2/b_1)} k_t - c_t. \tag{46}$$

Linearizing (45) and (46) we have

$$\begin{bmatrix} \dot{c}_t \\ \dot{k}_t \end{bmatrix} = \begin{bmatrix} -\dfrac{\gamma b_2}{b_1} & \dfrac{\beta \gamma c^*}{\sigma(b_1 + \beta k^*)} \\ -\dfrac{\sigma b_2}{b_1} - 1 & \dfrac{\gamma}{b_1 + \beta k^*} \end{bmatrix} \begin{bmatrix} c_t - c^* \\ k_t - k^* \end{bmatrix}. \tag{47}$$

The trace and the determinant of the Jacobian matrix are given by

$$\mathrm{tr} = -\frac{\gamma}{b_1}\left(b_2 - \frac{b_1}{b_1 + \beta k^*}\right), \quad \det = \left(\frac{\gamma}{b_1 + \beta k^*}\right)^2 \left(\frac{\beta}{\sigma}k^* - b_2\right). \tag{48}$$

Note that indeterminacy requires the determinant to be positive, so that $\beta = -\rho_Y > 0$, which in turn implies that $\gamma > 0$ to have positive values of the discount rate. We thus have the following result on indeterminacy:

Proposition 3.2. *If, evaluated at a steady state,*

$$\frac{\beta}{\sigma}k^* - b_2 > 0, \tag{49}$$

$$b_2 - \frac{b_1}{b_1 + \beta k^*} > 0, \tag{50}$$

then the equilibria near the steady state are indeterminate.

The steady-state solutions for k_t can be solved from the equation, $b_1 + \beta k = \Delta k^{b_2/(b_1+b_2\sigma)}$, where $\Delta = \gamma/(Ab_2^{b_2})^{1/(b_1+b_2\sigma)} > 0$. If $\beta < 0$, then there exists only one steady state, which, from (48), is a saddle point. On the other hand, it is clear from (49) and (50) that $\beta > 0$ ($\rho_Y < 0$) is a necessary condition for the equilibrium to be indeterminate, which says that the agent becomes more patient as the economy's living standard increases. If $\beta > 0$, then there normally exist two steady-states, of which the smaller one is a saddle point, whereas the larger one is indeterminate. The result is similar to that in Example 2.3 of Section 2. A numerical example is, $\sigma = 0.95$, $b_1 = 0.33$, $b_2 = 0.67$, $\beta = 0.0035$, $\gamma = 0.05$, $A = 1.075$.

Notice that under the necessary condition for indeterminacy, $\beta > 0$, the signs for the elements of the Jacobian matrix in (47) are given by

$$\begin{bmatrix} - & + \\ - & + \end{bmatrix}, \tag{51}$$

which has the same form as the models in Section 2 and the previous subsection (see (33)). This happens because in the employment efficiency condition (44), labor can be expressed as a decreasing function of c_t and an increasing function of k_t, so does the reduced form of the individual income (or output) in the second term in the bracket of the Euler equation (45). Thus the model operates as though the discounting function increases in average consumption and decreases in average capital through the dependence of the discounting function on average income, mimicking the situation in Section 2.

Benhabib and Farmer (1994) present a one-sector model with endogenous labor supply and with a constant discount rate. The indeterminacy result in their model relies on technological increasing returns to scale. A significant difference of the mechanism for indeterminacy in this subsection with theirs is that here the labor demand curve is downward-sloping, whereas the labor demand curve is upward-sloping in their model. In addition, contrary to Eq. (44), under the necessary condition (or the labor market condition) for indeterminacy in Benhabib and Farmer (1994) labor, and in turn, the reduced form of the social production, can be expressed as an increasing function of c_t and a decreasing function of k_t. Thus, in contrast to (51), indeterminacy happens in their setup when the Jacobian matrix takes the form,[16]

$$\begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix} = \begin{bmatrix} + & - \\ + & - \end{bmatrix}. \tag{52}$$

As is also noted in the literature (Aiyagari, 1995; Wen, 2001), indeterminacy happens in Benhabib and Farmer (1994) as under the labor market condition in their paper the wage rate (labor income) is increasing in labor input. In such a case since consumption depends on permanent income and an increase in the initial consumption level can induce a higher permanent income, agents' expectations can be self-fulfilling. In contrast, in the models of this paper, multiple equilibrium paths become possible because *the private marginal product or the rental rate of capital is increasing in capital* (as in the setup of the previous subsection) or in mechanisms that mimic such a relationship (as in the setups of this subsection and Section 2), and thus $j_{12} > 0$ in these models. In the Benhabib and Farmer model, the sign of the element j_{12} in the Jacobian matrix (52) does not change relative to the benchmark Ramsey–Cass–Koopmans model. This holds as in their model in its reduced form the private marginal product of capital is decreasing in capital, just as in the Ramsey–Cass–Koopmans model. Interestingly, the destabilizing factor,

[16]The Jacobian matrix in the model with endogenous labor supply and capital utilization in Wen (1998) has the same property as that of the Benhabib and Farmer model.

i.e., $j_{21}>0$ in (52), in the Benhabib and Farmer model is similar to that of the model in Kehoe et al. (1992) with productive consumption (see the corresponding element in the Jacobian matrix (36)), and the stabilizing factor, i.e., $j_{22}<0$ in (52), is similar to that of the model in Kehoe (1991) with a negative capital externality (see the corresponding element in the Jacobian matrix (35)). Both the latter two models have exogenous labor supply.

Finally, we note that the result in this subsection is robust to the more general preference specification with finite labor elasticities, that is, when the felicity is given by, $u(c_t, l_t) = (1-\sigma)^{-1}c_t^{1-\sigma} - (1+\chi)^{-1}l_t^{1+\chi}$, where $\chi \geqslant 0$. It can be verified that local and global indeterminacy can happen for reasonably large value of $\chi > 0$ (e.g., $\chi = 5$, $\sigma = 0.9$, $b_1 = 0.3$, $b_2 = 0.7$, $\beta = 0.0075$, $\gamma = 0.0043$, $A = 0.9$).

However, for sufficiently large χ local indeterminacy cannot happen. The reason is that with the general felicity, the labor efficiency condition becomes, $l_t = c_t^{-\sigma/(b_1+\chi)}(b_2 A k_t)^{b_1/(b_1+\chi)}$, and the dependence of l_t and hence y_t on c_t will be weaker as χ becomes larger, which is essential for the indeterminacy result. Indeed, the dependence of l_t or y_t on c_t will disappear completely with inelastic labor supply (i.e., $\chi = \infty$), in which case, the discount rate depends only on k_t so that indeterminacy is not possible, as is already known from the results in Section 2.

4. Impatience and endogenous growth

In this section we extend the analysis to a setup with endogenous growth. For specific discounting and felicity functions, we show below that an indeterminate balanced growth path (BGP) can arise, and in some cases local indeterminacy can be associated with global indeterminacy. The model below presents an alternative of the recent models such as Dolmas (1996), Drugeon (1996a, b) and Palivos et al. (1997) which incorporate both endogenous time preference and unbounded growth.

As is well known, for the case with a constant discount rate nondecreasing returns-to-scale technology is required for the existence of unbounded growth. We thus consider the linear technology.[17] We first establish the following lemma.

Lemma 4.1. *Given the linear technology* (Ak) *as well as the discounting function* $\rho(C, Y) > 0$ *and the felicity* $u(c) > 0$, *if a BGP exists, then* (i) *in order for* ρ *to be invariant at the BGP*, $\rho(C, Y)$ *must be homogenous of degree zero in* C *and* Y, *and* (ii) *under* (i) *the felicity is the isoelastic function.*[18]

Proof. See the Appendix. □

[17] In the context of the endogenous growth literature, linear technology can be obtained by aggregating physical and human capital into a composite good in a two-sector economy that follows a steady-state path, has no technological externalities, and has identical technologies in the output and human capital accumulation sectors.

[18] Palivos et al. (1997, pp. 220–221) use a specification where the discounting function is homogenous of degree zero in c and K, whereas Drugeon (1996a, Section III) considers one where the discounting function is homogenous of degree zero in c and k.

Note that the model here contains as a special case the celebrated endogenous growth model with the linear technology and a constant discount rate (see, e.g., Rebelo, 1991). Let the discounting function and the felicity be $\rho = \rho(C/Y)$, and $u = (1-\sigma)^{-1}c^{1-\sigma}$, where $0<\sigma<1$. Solving the representative agent's problem yields

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left[A - \rho\left(\frac{c_t}{Ak_t}\right)\right], \tag{53}$$

$$\frac{\dot{k}_t}{k_t} = A - \frac{c_t}{k_t}. \tag{54}$$

We introduce a new variable, ω_t, to denote the consumption–income ratio, that is,

$$\omega_t = \frac{c_t}{Ak_t}. \tag{55}$$

Then we have (ignoring time indexes)

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k} = \frac{1}{\sigma}[A - \rho(\omega)] - (A - A\omega) \equiv h(\omega), \tag{56}$$

i.e.,

$$\dot{\omega} = \omega h(\omega). \tag{57}$$

Here ω is a jump variable without an initial condition. Denote by ω^* the steady-state value of ω. If $h'(\omega^*)<0$, then the solutions for ω near ω^* are indeterminate, which, from Eqs. (53) and (54), implies that the growth rates for c_t and k_t are indeterminate. At the steady state ω^* (or the BGP), the growth rates for c and k are equal. We summarize the result in the following proposition:

Proposition 4.1. *If, evaluated at a steady state $\omega^*>0$, $h'(\omega^*)<0$, then for ω near ω^* there exist a continuum of equilibrium values for the growth rates $\dot{c}/c$ and $\dot{k}/k$.*

It is clear that indeterminacy is not possible if the discount rate ρ is a constant, as in that case $h' \equiv A>0$. Generally if $h'>0$ at a steady-state value of ω, then locally there exists no transitional dynamics and the only solution is the BGP itself. As $h' = A - (1/\sigma)\rho'$, a necessary condition for local indeterminacy is that $\rho'(\omega^*)>0$, which in turn implies that $\rho_C(\omega^*)>0$ and $\rho_Y(\omega^*)<0$. Therefore, that the discount rate increases with average consumption and decreases with average income is again necessary for indeterminacy, just as in the case with bounded growth in Section 2.

We next provide two simple examples. In the first example (Example 4.1) there exists at most one BGP, which can be locally indeterminate, whereas in the second example (Example 4.2) multiple (two) steady states can arise so that there can be both local and global indeterminacy.

Example 4.1. Let the discounting function be linear in C/Y, i.e.,

$$\rho = p + q \times (C/Y) \quad \text{where } p,\ q \text{ are parameters.} \tag{58}$$

Then Eq. (57) becomes

$$\dot{\omega} = \omega\left[A\left(\frac{1}{\sigma} - 1\right) - \frac{p}{\sigma} - \left(\frac{q}{\sigma} - A\right)\omega\right]. \tag{59}$$

There exists at most one positive steady-state solution to (59), so that there exists at most one BGP. Indeterminacy requires that $q/\sigma - A > 0$, and hence $q > 0$ is necessary for indeterminacy. In addition, to have positive steady-state value for ω, $A(1/\sigma - 1) - p/\sigma > 0$ must be satisfied. Note that $q > 0$ implies that $\rho_C > 0$ and $\rho_Y < 0$.

Example 4.2. Let the discounting function be

$$\rho = m + n \times (C/Y)^{-1} \quad \text{where } m,\ n \text{ are parameters.} \tag{60}$$

Eq. (57) now becomes

$$\dot{\omega} = A\omega^2 + \left[\left(\frac{1}{\sigma} - 1\right)A - \frac{m}{\sigma}\right]\omega - \frac{n}{\sigma} \equiv z(\omega). \tag{61}$$

The RHS of Eq. (61) is a U-shaped parabola, and there may exist one or two (or no) positive steady-state solutions depending on the parameter values. If $n > 0$, then there exist two solutions to $z(\omega) = 0$, which have opposite signs. Ruling out the negative root, for the positive root we have $z' > 0$, so that the only solution close to the positive root is the root itself. In that case, the growth rates $\dot{c}/c$ and $\dot{k}/k$ are unique. On the other hand, if $n < 0$, then there may exist two positive steady states. When there are two positive steady states (and hence two BGPs), the smaller one is indeterminate, and the larger one is determinate. More formally, there exist two positive steady states if, in addition to the requirement that $n < 0$, the parameters satisfy the conditions: (i) $[(1/\sigma - 1)A - m/\sigma]^2 + 4nA/\sigma > 0$, and (ii) $(1/\sigma - 1)A - m/\sigma < 0$. In fact, since $z(\omega)$ is a parabola, if there exists a positive steady state which is indeterminate, there must exist another, larger one which is determinate. Here as in Example 4.1, $n < 0$ implies that $\rho_C > 0$ and $\rho_Y < 0$.

Note that from (54), at the BGP(s), the growth rates $\dot{c}/c$ and $\dot{k}/k$ are given by $A(1 - \omega^*)$. Thus in this example when there are two BGPs, the one with the higher growth rate corresponds to the lower steady-state value of the consumption – income ratio ω and is locally indeterminate, and the other one with the lower growth rate corresponds to the higher steady-state value of ω and is locally determinate. We have thus both local and global indeterminacy: given an initial condition it is not possible to choose initial value of c to either place the economy in the determinate BGP with lower growth rate or in a neighborhood the indeterminate BGP with higher growth rate. These results are similar to those obtained in Benhabib and Perli (1994) in a Lucas-type, two-sector endogenous growth model with endogenous labor supply. A simple numerical example with two BGPs is, $m = 0.025$, $n = -0.001$, $A = 0.025$, $\sigma = 0.8$, in which the two steady-state values for ω are $\omega_1^* = 0.95$ and $\omega_2^* = 0.05$, and the corresponding growth rates for $\dot{c}/c$ and $\dot{k}/k$ at the BGPs are 0.00125 (lower) and 0.02375 (higher).

5. Conclusion

This paper considers the hypothesis that individual time preference is socially determined, in that it depends on some aggregate economic variables. We study the implications of the hypothesis for the aggregate dynamics in extended one-sector neoclassical growth models. We find that equilibrium indeterminacy can easily arise. These results are in sharp contrast to those obtained in models with a constant discount rate or models with the discount rate depending on variables taken as internal by the agent.

For analytical convenience, we limit the analysis to the case that is the polar opposite of the traditional models of endogenous time preference by assuming that the discount rate depends exclusively on social variables taken as external by the agent. In addition to the external factors, allowing also for dependence of the discount rate on variables taken as internal by the agent – such as the agent's own consumption and income – would be of more interest. That is, one could consider the following general specification for the discounting function:

$$\rho = \rho(\text{individual consumption and/or income, average consumption and or income}),$$

which contains as special cases the models in this paper as well as the Uzawa–Epstein type models. But that would introduce an additional state variable and would thus considerably complicate the analysis by increasing the system's dimension. However, with the continuity assumption for relevant functions the results obtained in this paper would be consistent with those from a more general analysis as long as the influence of the social forces is, on the whole, significantly more important than that of the individual or nonsocial factors.[19]

Finally, to examine the empirical plausibility of the conditions for indeterminacy in the models of this paper, in addition to the evidence reviewed in the introduction, further empirical research may test the particular time preference specifications of this paper against the standard assumption of a constant discount rate. Ogawa (1993) has already made some progress along this line by estimating the discounting functions of three economies (Taiwan, Korea, and Japan) using the time-series data. The time preference specifications of the present paper may also be estimated by following the econometric method suggested in Farmer and Guo (1995), who use lagged values of the endogenous variables as instruments in a model with multiple equilibria in the presence of production externalities. In addition, as a further extension of the present paper's analysis, one could by introducing sunspot shocks calibrate discrete-time versions of the models with bounded growth in this paper (especially the model with elastic labor supply in Section 3.2) so as to evaluate, in comparison with the existing endogenous business-cycle models (e.g., Wen, 1998; Schmitt-Grohé, 2000; Benhabib and Wen, 2004), whether their predictions fit the actual macroeconomic data.

[19]For examples of models with the discount rate depending on both individual and social variables, see, Drugeon (1996a, 1998).

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Appendix A

We prove Lemma 4.1 in this appendix. From the agent's maximization problem, we have

$$\frac{\dot{c}}{c} = \frac{u'(c)}{c \cdot u''(c)}(\rho(c, y) - A), \tag{A.1}$$

$$\frac{\dot{k}}{k} = A\left(1 - \frac{c}{y}\right), \tag{A.2}$$

where $y = Ak$. From Eq. (A.2), if a finite, nonzero BGP exists, then c/y is a constant at the BGP. By assumption ρ is invariant at the BGP, that is, $\rho(c, y) = \Omega$ (a positive constant). Since $(\varepsilon c, \varepsilon y)$ is also at the BGP for any $\varepsilon > 0$, then $\rho(\varepsilon c, \varepsilon y) = \Omega$. Thus, $\rho(c, y)$ is homogeneous of degree zero in c and y, i.e., $\rho(C, Y)$ must be homogeneous of degree zero in C and Y. At the BGP, $\dot{c}/c$ is a constant, then from Eq. (A.1) we have

$$\frac{u'(c)}{c \cdot u''(c)} = \text{constant}. \tag{A.3}$$

Solving the differential equation for $u(c)$ in (A.3), we have that the felicity is of the CES form.

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