Preferences, endogenous discount rate, and indeterminacy in a small open economy model

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Abstract

This paper characterizes indeterminacy conditions for a small open economy business cycle model. The preferences are the utility function commonly used in closed-economy models in the literature, coupled with an endogenously determined discount rate. It shows that indeterminacy can occur under arbitrarily small external effects, independently of the intertemporal elasticities in consumption and labor supply.

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1. Introduction

It is well understood by now that under certain market imperfection conditions models of business cycle can be subject to indeterminacy. Indeterminacy means that from the same initial condition there exist an infinite number of equilibria, all of which converge to a common steady state. The literature on indeterminacy highlights the importance that equilibria need not be uniquely determined by the economy’s fundamentals, and that the existence of indeterminate equilibria is associated with the possibility of self-fulfilling prophecies.¹

Most models in the literature are closed-economy, and focus on the empirical plausibility of the conditions for indeterminacy. While early models relied on large increasing returns or external effects to

¹ For a recent extensive survey of the literature, see, Benhabib and Farmer (1999).
generate indeterminacy, recent research, in line with empirical estimates on technologies, demonstrates that only small market imperfections are needed.\(^2\) The indeterminacy conditions also have to do with preferences. In particular, most papers adopt the instantaneous utility function from the RBC literature, i.e., \(u(c_t, l_t) = (1 - \sigma)\left(1 - (1 - \sigma) - 1 - (1 + \chi)^{-1}\right)\), and find that to have indeterminacy under small market imperfections, the intertemporal elasticities in consumption or labor supply must be sufficiently large.

This paper extends research on indeterminacy to a small open economy business cycle model. To do so, we use technologies with social constant returns and private decreasing or constant returns, similar to those in Benhabib and Nishimura (1998), and characterize the indeterminacy conditions under the above instantaneous utility function used in the closed economy. This is a necessary and useful extension as most economies in the world are small and open.\(^3\)

We also follow the literature of small open economy RBC models by incorporating into the model an endogenously determined discount rate. Under such a preference specification, we show that indeterminacy occurs for technologies with arbitrarily small externalities, independently of the intertemporal elasticities in consumption and labor supply—a result that is in contrast to those obtained in closed-economy business cycle models.

The assumption of an endogenous discount rate is crucial for obtaining the indeterminacy result for the small open economy, because it eliminates the zero-eigenvalue problem that would exist with a constant discount rate. This makes it possible to identify conditions for indeterminacy, which generally require that the Jacobian matrix be nonsingular.

2. The two-sector open economy

2.1. The model

Consider a small open economy inhabited by an infinite-lived representative agent who maximizes the intertemporal utility function

\[
\int_0^\infty \left( \frac{1}{1 - \sigma} c_t^{1-\sigma} - \frac{1}{1 + \chi} l_t^{1+\chi} \right) e^{-\int_0^t \rho(\tilde{c}_s) ds} dt
\]

where \(c_t\) is consumption of traded goods, \(l_t\) is labor supply. Here \(\sigma > 0\), and \(\chi > 0.\(^4\) Following Shi (1999) and Schmitt-Grohe and Uribe (2002), we assume that the instantaneous discount rate (or rate of time

\(^2\) For an example of early models, see Benhabib and Farmer (1994); for an example of recent models, see Benhabib and Nishimura (1998).
\(^3\) This paper is also a realistic extension of small open economy models of Weder (2001) and Meng and Velasco (in press), which assume inelastic labor supply.
\(^4\) We ignore the knife-edge cases when \(\sigma = 0\) or/and \(\chi = 0\). The result in this paper can also be extended to the case with a general utility function, \(u(c) - \nu(l)\), where \(u(\cdot)\) is strictly concave, and \(\nu(\cdot)\) strictly convex.
preference) is of modified Uzawa type. In particular, it is strictly positive, and is an increasing function of the economy-wide average consumption, i.e.,

$$\rho' (\bar{c}_s) > 0$$  \(2\)

The implication of Eq. (2) is that a higher level of the average consumption at time \(s\) increases the discount rate to the agent’s utility at and after \(s\).\(^5\) The average consumption in the discount rate captures the “jealousy” (or “admiration”) effect of consumption externalities, recently emphasized among other areas in the literature on asset pricing.\(^6\)

The economy is open to full international capital mobility, so that the agent has access to net foreign bonds \(d_t\), denominated in units of consumption goods, that pay an exogenously given world interest rate \(r\). Assume that consumption goods are tradeable and capital goods non-tradeable.

On the production side, there are two sectors: one producing the tradeable consumption goods \(y_{1t}\) and the other the non-tradeable investment goods \(y_{2t}\). The production functions are assumed to be Cobb–Douglas with factor input generated externalities, i.e.,

$$y_{1t} = \frac{\rho_{y_{1}}}{l_{1t}} k_{1t} l_{1t}^{a_{10}} \prod_{1}^{a_{11}}$$

where

$$y_{2t} = \frac{\rho_{y_{2}}}{l_{2t}} k_{2t} l_{2t}^{a_{20}} \prod_{2}^{a_{21}}$$

$$k_{1t} + k_{2t} = k_t; \quad l_{1t} + l_{2t} = l_t$$

Here \(l_{1t}\) and \(k_{1t}\) denote the capital and labor services used by the individual firm in the consumption good producing sector, and \(l_{2t}\) and \(k_{2t}\) for the investment good producing sector. \(k_t\) is the aggregate capital stock. The production functions satisfy the following assumption.

**Assumption 1.** The technologies in Eq. (3) exhibit social constant returns to scale, and private decreasing or constant returns to scale, that is,

$$\alpha_0 + \alpha_1 + \alpha_0 = 1; \quad \alpha_0 + \alpha_1 \leq 1, \alpha_0 > 0, \alpha_1 > 0;$$

$$\beta_0 + \beta_1 + b_0 + b_1 = 1; \quad \beta_0 + \beta_1 \leq 1, \beta_0 > 0, \beta_1 > 0;$$

Under Assumption 1, the technologies can exhibit either private decreasing or constant returns to scale. Private constant returns can happen if we allow for negative externalities and if \(a_0 = -a_1(b_0 = -b_1)\). In the case of private decreasing returns, since firms earn positive profits, a fixed entry cost is required to deter new entrants.\(^7\)

\(^5\) While Shi (1999) considers an endowment type economy, Schmitt-Grohé and Uribe (2002) study an RBC model with endogenous labor supply. Unlike Schmitt-Grohé and Uribe who postulate that the discount factor is a function of the average instantaneous utility, the present paper assumes that the discount rate depends only on the average consumption and not on the average labor supply.

\(^6\) See, for example, Abel (1990) and Campbell and Cochrane (1999). However, these papers put the average consumption in the instantaneous utility function. In the present paper, preferences exhibit jealousy (admiration) if the instantaneous utility is positive (negative).

\(^7\) The production functions are similar to those in Benhabib and Nishimura (1998). However, since we allow for negative factor input externalities, indeterminacy is possible under both private and social constant returns—see Drugeon (2000) about this point in a closed-economy model.
The agent’s budget constraint is

\[ \dot{d}_t = rd_t + y_{1t} + pt_1y_{2t} - c_t - p_1i_t \]

where \( p_t \) is the relative price of the nontraded goods to the traded goods. \( i_t \) denotes gross investment, so that the law of motion for capital is

\[ \dot{k}_t = i_t - \delta k_t \]

Eqs. (5) and (6) can be consolidated into

\[ \dot{z}_t = rz_t + y_{1t} + p_t y_{2t} - c_t + k_t(\dot{p}_t - rp_t - \delta p_t) \]

where \( z_t = d_t + p_t k_t \). The agent’s problem is to choose \( c_t, l_{1t}, l_{2t}, i_t, k_{1t}, k_{2t} \) and \( d_t \) to maximize Eq. (1), subject to (Eqs. (3), (4) and (7), and given \( k_0 \) and \( d_0 \). The present-value Hamiltonian is

\[ H = \left( \frac{1}{1 - \sigma} c_t^{1-\sigma} - \frac{1}{1 + \chi} l_t^{1+\chi} \right) e^{-\int_0^t \rho(c_s)ds} + \phi_t [rz_t + y_{1t} + p_t y_{2t} - c_t + k_t(\dot{p}_t - rp_t - \delta p_t)] \\
+ \mu_t (k_t - k_{1t} - k_{2t}) + \omega_t (l_t - l_{1t} - l_{2t}) \]

where \( \phi_t \) is a costate variable, \( \mu_t \) and \( \omega_t \) are the rental rate of capital goods and the wage rate of labor, all in terms of the consumption goods. In solving the problem, the agent takes the average consumption, \( \bar{c}_t \), as given, but in equilibrium \( \bar{c}_t = c_t \). First-order conditions are (denoting \( \alpha_0 + \alpha_1 = \alpha; \beta_0 + \beta_1 = \beta \)),

\[ c_t^{-\sigma} e^{-\int_0^t \rho(c_s)ds} - \phi_t = 0 \]

\[ l_t^{\alpha_1} e^{-\int_0^t \rho(c_s)ds} = \phi_t \beta_0 l_{1t}^{\beta_0 - 1} k_{1t}^{1-\beta} \]

\[ \mu_t = \phi_t \beta_1 l_{2t}^{\beta_1} k_{2t}^{-\beta} \]

\[ \omega_t = \phi_t \beta_0 l_{1t}^{\beta_0 - 1} k_{1t}^{1-\beta} = \phi_t \beta_1 l_{2t}^{\beta_1} k_{2t}^{-\beta} \]

\[ r \phi_t = -\dot{\phi}_t \]

\[ \dot{p}_t = p_t (r + \delta - \beta_1 l_{2t}^{\beta_1} k_{2t}^{-\beta}) \]
In addition, the market-clearing condition for the nontraded goods and the current account are:

\[ \dot{k}_t = y_{2t} - \delta k_t \]  

(14)

\[ \dot{d}_t = rd_t + y_{1t} - c_t. \]  

(15)

2.2. The indeterminacy result

By introducing a new costate variable, \( \lambda_t = \phi \rho \frac{d}{d s} \), the first-order conditions (Eqs. (8), (9) and (12)) become, respectively,

\[ c_t^{-\sigma} = \lambda_t \]  

(16)

\[ I_t^\lambda = \lambda_t \beta_0 p_t l_2t^{1-\beta} k_2t^{1-\beta} \]  

(17)

\[ \dot{\lambda}_t = \lambda_t [\rho(c_t) - r] \]  

(18)

From Eqs. (16) and (18), we have

\[ \dot{c}_t = \frac{c_t}{\sigma} [r - \rho(c_t)] \]  

(19)

In the Appendix A, we derive the dynamic equations for \( p_t, k_t \) and \( d_t \),

\[ \dot{p}_t = p_t (r + \delta - \beta_1 g(p_t)^\beta) \]  

(20)

\[ k_t = \left[ \frac{z_0 \beta_1 g(p_t)^\beta}{z_0 \beta_1 - z_0 \beta_0} - \delta \right] k_t - \left( \frac{z_1 \beta_0}{z_0 \beta_1 - z_1 \beta_0} \right) \left[ \beta_0 c_t^{-\sigma} p_t g(p_t)^{\beta-1} \right]^{1/\chi} \]  

(21)

\[ \dot{d}_t = rd_t + y_{1t}(p_t, k_t, c_t) - c_t \]  

(22)

where, \( g(p_t) = \xi p_t^{1/(\alpha + a_0)(\beta_1 + b_1)-(z_1 + a_1)(\beta_0 + b_0)} \), \( \xi \) is a positive parameter, and \( y_{1t}(p_t, k_t, c_t) \) is given in the Appendix A.

The dynamic system consists of four differential equations (Eqs. 19–22) for \( (c_t, p_t, k_t, d_t) \). This is in contrast to closed-economy models in the literature that are generally associated
with a system of two differential equations. Linearizing around the unique steady state, we obtain

\[
\begin{pmatrix}
\dot{c}_t \\
\dot{p}_t \\
\dot{k}_t \\
\dot{d}_t
\end{pmatrix} = 
\begin{pmatrix}
\frac{c^* \rho'(c^*)}{\sigma} & 0 & 0 & 0 \\
0 & j_{22} & 0 & 0 \\
j_{31} & j_{32} & j_{33} & 0 \\
j_{41} & j_{42} & j_{43} & r
\end{pmatrix}
\begin{pmatrix}
c_t - c^* \\
p_t - p^* \\
k_t - k^* \\
d_t - d^*
\end{pmatrix}
\]

The four eigenvalues of the Jacobian matrix are, \(-\frac{c^* \rho'(c^*)}{\sigma} < 0, r > 0\), and

\[
j_{22} = \frac{\beta(r + \delta)}{(\alpha_1 + a_1)(\beta_0 + b_0) - (\alpha_0 + a_0)(\beta_1 + b_1)}
\]

\[
j_{33} = \frac{\alpha_0 r + \alpha_0 \delta (1 - \beta_1) + \alpha_1 \beta_0}{\alpha_0 \beta_1 - \alpha_1 \beta_0}
\]

Here physical capital \(k_t\) and bonds \(d_t\) are predetermined variables and evolve continuously, while \(p_t\) and \(c_t\) are jump variables. Indeterminacy requires both \(j_{22}\) and \(j_{33}\) to be negative. We thus have the following indeterminacy result.

**Proposition.** If the nontraded capital goods sector is labor intensive from the private perspective \((j_{22} < 0)\) and capital intensive from the social perspective \((j_{33} < 0)\), then there exist a continuum of equilibrium paths converging to the steady state.

Note that without externalities \(j_{22}\) and \(j_{33}\) are of opposite signs, so that there are two negative and two positive eigenvalues, and the equilibrium path is unique. With small externalities, indeterminacy can occur under the factor intensity conditions given in the proposition.

It is clear from the proposition that indeterminacy can arise under arbitrarily small externalities. Moreover, the indeterminacy condition is independent of the intertemporal elasticities in consumption and labor supply. The intuition for this result is straightforward. In the open economy, the curvature of the utility function does not affect the investment decision, since unlike in the closed economy the agent can always borrow from the outside world to finance his consumption. The above indeterminacy result is in contrast to the two-sector closed-economy indeterminacy result in Benhabib and Nishimura (1998), which requires the extreme assumption of linear or close-to-linear utility.

It is important to note the role that the endogenous discount rate plays for the indeterminacy result. If the discount rate were constant, there would exist a zero-eigenvalue and the Jacobian matrix is singular.
as the standard assumption is to let the discount rate be equal to the world interest rate). Then we would not be able to obtain an indeterminacy condition, even if such a condition existed at all.

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**Appendix A**

Dividing Eq. (10) by Eq. (11) yields

\[
\frac{\alpha_1}{\alpha_0} \frac{l_{1t}}{k_{1t}} = \frac{\beta_1}{\beta_0} \frac{l_{2t}}{k_{2t}}
\]  

(A.1)

Using Eqs. (10) and (A.1) to solve for \((l_{2t})(k_{2t})\), we have

\[
\frac{l_{2t}}{k_{2t}} = \frac{\zeta p_t^{-\frac{1}{\beta}}}{\xi p_t - \frac{\beta_1}{\beta_0} \frac{l_{2t}}{k_{2t}}} = g(p_t)
\]  

(A.2)

where \(\zeta = (\beta_1/\alpha_1)(\alpha_1 \beta_0/\alpha_0 \beta_1)^\gamma\). Substituting Eq. (A.2) into Eq. (13) yields the dynamic equation (Eq. (20)) for \(p_t\). From Eqs. (4), (A.1), (A.2) and (9), we can solve for \(k_{2t}\):

\[
k_{2t} = \left(\frac{\alpha_0 \beta_1}{\alpha_0 \beta_1 - \alpha_1 \beta_0}\right)^\beta k_t - \frac{\alpha_1 \beta_0}{\alpha_0 \beta_1 - \alpha_1 \beta_0} \frac{[\beta_0 c_t^{-\sigma} p_t g(p_t)^{\beta-1}]^{1/\gamma}}{g(p_t)}
\]  

(A.3)

Thus, we have

\[
y_{2t} = k_{2t} \left(\frac{l_{2t}}{k_{2t}}\right)^\beta = \left[\frac{\alpha_0 \beta_1}{\alpha_0 \beta_1 - \alpha_1 \beta_0}\right]^\beta k_t - \left(\frac{\alpha_1 \beta_0}{\alpha_0 \beta_1 - \alpha_1 \beta_0}\right) \frac{[\beta_0 c_t^{-\sigma} p_t g(p_t)^{\beta-1}]^{1/\gamma}}{g(p_t)}
\]  

(A.4)

From the market-clearing condition (Eq. (14)), we obtain Eq. (21). Similarly, we can compute \(y_{1t}\) (and hence the dynamic equation for \(d_t\)):

\[
y_{1t} = k_{1t} \left(\frac{l_{1t}}{k_{1t}}\right)^\gamma = \left\{\left(\frac{-\alpha_1 \beta_0}{\alpha_0 \beta_1 - \alpha_1 \beta_0}\right) k_t + \frac{\alpha_1 \beta_0}{\alpha_0 \beta_1 - \alpha_1 \beta_0} \frac{[\beta_0 c_t^{-\sigma} p_t g(p_t)^{\beta-1}]^{1/\gamma}}{g(p_t)}\right\}
\]  

\[
\times \left[\frac{\alpha_0 \beta_1}{\alpha_1 \beta_0} g(p_t)\right]^{\gamma}
\]  

(A.5)
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