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Economic Theory

Investment, interest rate rules, and equilibrium determinacy*

Qinglai Meng and Chong K. Yip

Department of Economics, Chinese University of Hong Kong, Shatin, HONG KONG
(email: {meng2000,chongkeeyip}@cuhk.edu.hk)

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Summary. By adding endogenous investment to a flexible-price, money-in-the-utility-function model, this paper studies the role that physical capital plays in stabilizing the real side of the economy when the monetary authority follows interest-rate feedback rules. We show that with inelastic labor supply equilibrium uniqueness is ensured under both active and passive monetary policies. For the case where money affects both preferences and technology, the uniqueness result remains true under active monetary policy. With endogenous labor supply, the uniqueness result holds again regardless of the stance of monetary policies for the case with separable leisure, but indeterminacy remains likely under both active and passive monetary policies when leisure is nonseparable.

Keywords and Phrases: Investment, Interest-rate feedback rules, Determinacy versus indeterminacy.

JEL Classification Numbers: E52, O42.

1 Introduction

Recently there has been a growing interest in studying the link between interest-rate feedback rules and macroeconomic stability. A well-known result obtained in the literature is that an active monetary policy, that is, a policy that responds to a one percentage point increase in inflation with a more than one percentage point increase in the nominal interest rate, is stabilizing by ensuring the uniqueness of equilibrium of the real side of the economy. At the same time, a passive monetary

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Correspondence to: C.K. Yip

policy, where a one percentage point increase in inflation is met with less than one percentage point increase in the nominal interest rate, destabilizes the economy by generating indeterminacy and expectations-driven fluctuations.[1]

Some authors have recently started to challenge the above result. For example, in Benhabib et al. [2], they show that whether a particular monetary policy is conducive to macroeconomic stability depends crucially on the way in which money is assumed to enter preferences and technology. In particular, in the context of both flexible-price and sticky-price models where money affects both preferences and production possibilities, they show that under certain conditions an active monetary policy does not necessarily bring about the determinacy of equilibrium, and that a passive monetary policy may do so. By appending endogenous investment to a benchmark continuous-time sticky-price model, Dupor [6] adds a further twist to the early standard results by showing that only passive monetary policy can generate uniqueness of equilibrium, and that active monetary policy leads to either equilibrium indeterminacy or no equilibrium at all.

In view of the fact that the results in the literature are obtained in both flexible and sticky price models, in this paper we add endogenous investment to the neoclassical flexible-price models studied by prior authors such as Leeper [9], Benhabib et al. [2] and Schmitt-Grohé and Uribe [3]. The rationale of including investment is overwhelming since investment is a significant fraction of GDP and it is much more volatile than consumption among the GDP components.[2] In addition, Benhabib et al. [2] suggest that \it is desirable for future research aimed at evaluating the stabilizing properties of alternative monetary-policy rules to take into account the supply-side channel for the transmission of nominal interest-rate variations." (p. 185). In this paper we explore how the inclusion of endogenous investment affects the dynamics and stability of the real side of the economy.

We first study the dynamics under interest-rate feedback rules in the flexible-price monetary growth model with endogenous investment and with inelastic labor supply. We find that under general assumptions about preferences and technology equilibrium uniqueness is ensured under both active and passive monetary policies. This is in contrast to the results obtained by prior authors. The intuition for such a uniqueness result is straightfoward. With endogenous investment the dynamics of the monetary growth model usually reduces to a system of two differential equations. The addition of physical capital – a predetermined variable with a given initial condition – pins down the two by two system thereby guaranteeing uniqueness. In addition, with inelastic labor supply, we find that equilibrium uniqueness is obtained under the general interest-rate feedback rule that depends on both inflation and output.

We then study the robustness of the uniqueness result in two major extensions of the basic model. In the first extension, we follow Benhabib et al. [2] by introducing a demand for money by firms so that money affects not only preferences but also production possibilities. This specification of the model is based on the empirical fact that demand deposits of firms have been significantly more than those

[1] See, for example, Leeper [9], Rotemberg and Woodford [12], and Clarida et al. [5].

[2] Investment is about four times as volatile as consumption in the post-war U.S. economy.

of households in the U.S. for the past three decades.[3] The importance of allowing money to enter the production function is highlighted in a theoretical evaluation of interest-rate feedback rules in Benhabib et al. [2]. In particular, in their flexible-price model without investment, when this additional channel is included, they reverse the conventional results. They find that there is equilibrium uniqueness under passive monetary policy and indeterminacy under active monetary policy. We show that even if money affects both preferences and technology, with investment the uniqueness result remains true under active monetary policy, but may not hold under passive monetary policy.

The second extension that we consider is to allow for endogenous labor supply. We find that while the uniqueness result holds for the case with separable leisure again regardless the stance of monetary policies, equilibrium indeterminacy becomes likely under both active and passive monetary policies for the nonseparable leisure case. Indeterminacy with nonseparable leisure occurs because monetary policy can have important effects on production through its influence on labor supply.

The organization of the paper is as follows. The next section describes the basic money-in-the-utility-function model with neoclassical investment and with inelastic labor supply. Section 3 considers the extension where money is productive in the sense that it enters the neoclassical production function as a factor input. Section 4 studies the extension with endogenous labor supply for both separable and nonseparable leisure cases. Section 5 concludes.

2 A flexible-price monetary model with investment

In this section, we develop the basic flexible-price monetary growth model to study the dynamics of equilibrium under interest-rate feedback rules. Money enters the macroeconomy as an argument in the utility function of the representative household – the so-called money-in-the-utility-function (MIUF) approach.[4] By building a monetary growth model, we depart from Benhabib et al. [2] to allow for endogenous investment.

2.1 The economic environment

Households. The economy consists of a continuum of identical household-firms with unit mass, each of whom maximizes his lifetime utility according to

$$U = \int_0^\infty e^{-\rho t} u(c_t, m_t) dt \tag{1}$$

where ρ denotes the rate of time preference, c_t consumption, $m_t \equiv M_t/P_t$ real balances and M_t nominal money balances (held for nonproductive purposes), P_t

[3] See Mulligan [3] for more elaboration on this point.

[4] Our choice of the MIUF model is based on its generality. According to Feenstra [8], the MIUF model is functionally equivalent to the transactions-cost model. In addition, it has been shown that either the shopping-time model or cash-in-advance model can be rewritten as a MIUF model.

the nominal price level. The felicity function $u(.,.)$ is assumed to have the following properties:

Assumption 1. *The felicity function $u(.,.)$ is strictly increasing and strictly concave, and both its arguments c and m are normal goods, that is, $u_{cc}u_m - u_{cm}u_c < 0$, and $u_{mm}u_c - u_{cm}u_m < 0$.*

In addition to money the household can also hold nominal bonds and physical capital. The nominal bonds pay the nominal interest rate $R_t > 0$. The household's budget constraint is given by

$$c_t + \dot{m}_t + \dot{b}_t + \dot{k}_t = (R_t - \pi_t)\, b_t + y_t - \pi_t m_t - \tau_t \tag{2}$$

where k_t is the stock of capital, y_t the income from production, τ_t real lump-sum taxes, $\pi_t \equiv \dot{P}_t/P_t$ the inflation rate. Assume that the household-firm has access to a technology that transforms capital into final goods, and with the following properties:

Assumption 2. *The production function $y_t = f(k_t)$ is strictly increasing and strictly concave, that is, $f'(k_t) > 0 > f''(k_t)$.*

By defining the household's total real wealth as $a_t = m_t + b_t + k_t$, the household's budget constraint can be written as

$$\dot{a}_t = (R_t - \pi_t)\, a_t + y_t - (R_t - \pi_t)\, k_t - R_t m_t - c_t - \tau_t\ . \tag{3}$$

The household chooses sequences for c_t, m_t, k_t and a_t so as to maximize (1) subject to (3) and the standard no-Ponzi-game condition, taking as given a_0 and the time paths of τ_t, R_t and π_t. The optimality conditions associated with the household-firm's are given by

$$u_c\,(c_t, m_t) = \lambda_t \tag{4}$$

$$\frac{u_m\,(c_t, m_t)}{u_c\,(c_t, m_t)} = R_t \tag{5}$$

$$f'(k_t) = R_t - \pi_t \tag{6}$$

$$\dot{\lambda}_t = \lambda_t\,(\rho + \pi_t - R_t) \tag{7}$$

$$\lim_{t\to\infty} e^{-\int_0^{\infty}[R(s)-\pi(s)]ds} a_t = 0 \tag{8}$$

where λ_t is the costate variable associated with the household's instant budget constraint.

The Government. Following Benhabib et al. [2], we assume that monetary policy takes the form of an interest rate feedback rule whereby the nominal interest rate is set as a non-decreasing function of the inflation rate. Specifically,

$$R_t = R(\pi_t) \tag{9}$$

where $R(\pi_t)$ is continuous, non-decreasing and strictly positive, and there exists at least one $\pi^* > -\rho$ such that $R(\pi^*) = \rho + \pi^*$. The monetary policy is referred to as active if $R'(\pi^*) > 1$ and passive if $R'(\pi^*) < 1$ (assuming that $R'(\pi^*) \neq 1$). In addition, by assuming zero government spending at all times the government budget constraint can be written as

$$\dot{m}_t + \dot{b}_t = (R_t - \pi_t)\, b_t - \pi_t m_t - \tau_t \ . \tag{10}$$

For fiscal policies, following Benhabib, et al. [2], we can classify them into Ricardian fiscal policies and non-Ricardian fiscal policies. Ricardian fiscal policies are those that ensure that the present discounted value of total government liabilities converges to zero – that is, Eq. (8) is always satisfied - under all possible, equilibrium or off – equilibrium, paths of endogenous variables such as the price level, the money supply, inflation, or the nominal interest rate. For non-Ricardian fiscal policies, one example is that the government sets an exogenous path for lump-sum taxes, i.e., $\tau_t = \bar{\tau}$.

For the above monetary-fiscal regimes, one can demonstrate that price level determinacy (nominal determinacy) depends only on fiscal policy but not on monetary policy – a result that has been emphasized in the recent literature on the fiscal determination of the price level. In particular, one can show that if fiscal policy is Ricardian then the price level is indeterminate, and that under the non-Ricardian fiscal policy given above, the equilibrium displays price level determinacy (see, e.g., Benhabib et al. [2]). We show below, however, that the determinacy of real variables (real determinacy) is independent of the stance of both monetary and fiscal policies, and that there is always uniqueness of equilibrium.

2.2 Equilibrium dynamics

Equation (6) can be written as

$$f'(k_t) + \pi_t = R(\pi_t) \ . \tag{11}$$

One can solve Eq. (11) for π_t for equilibrium close to the steady state[5]

$$\pi_t = \pi(k_t) \tag{12}$$

where $\pi'(k_t) = -f''/(1 - R')$. By substitution c_t and m_t can be expressed as functions of λ_t and k_t

$$c_t = c(\lambda_t, R_t) = c(\lambda_t, k_t) \tag{13}$$

$$m_t = m(\lambda_t, R_t) = m(\lambda_t, k_t) \ . \tag{14}$$

In addition, we have

$$\partial c/\partial\lambda = (u_{mm} - R u_{cm})/\Delta < 0 \tag{15}$$

[5] Such a solution exists by the implicit function theorem since $R'(\pi^*) \neq 1$.

where $\Delta \equiv (u_{cc}u_{mm} - u_{cm}{}^2) > 0$.

It suffices to solve the following two dynamic equations

$$\dot{k}_t = f(k_t) - c(\lambda_t, k_t) \tag{16}$$

$$\dot{\lambda}_t = \lambda_t(\rho - f'(k_t)) \ . \tag{17}$$

Once solutions for k_t and λ_t are obtained from Eqs. (16) and (17), π_t and other variables are determined backward by using Eqs. (12) to (14). Linearizing Eqs. (16) and (17) at the steady state (λ^*, k^*), we have

$$\begin{pmatrix} \dot{\lambda}_t \\ \dot{k}_t \end{pmatrix} = A \begin{pmatrix} \lambda_t - \lambda^* \\ k_t - k^* \end{pmatrix}$$

where the Jacobian matrix A is given by

$$A = \begin{bmatrix} 0 & -\lambda^* f''(k^*) \\ -c_\lambda(\lambda^*, k^*) & f'(k^*) - c_k(\lambda^*, k^*) \end{bmatrix}$$

Since $\det(A) = -c_\lambda(\lambda^*, k^*)\lambda^* f''(k^*) < 0$, the Jacobian matrix A has two eigenvalues with opposite signs, and there is a unique saddlepath that converges to the steady state. Proposition 1 summarizes the result.

Proposition 1. *With inelastic labor supply, under both active and passive monetary policies* (*i.e., for both* $R'(\pi^*) > 1$ *and* $R'(\pi^*) < 1$)*, there exists a unique perfect foresight equilibrium in which all variables converge asymptotically to the st eady state.*

Note that in the absence of capital accumulation, local indeterminacy can occur easily – either when consumption and real money balances are Edgeworth complements and monetary policy is passive, or when consumption and real money balances are Edgeworth substitutes and monetary policy is active.[6] In the present case, the addition of endogenous physical capital as a predetermined variable eliminates the possibility of indeterminacy of equilibrium because it supplies an initial condition, which pins down the two-by-two dynamic system in terms of (λ_t, k_t).

It should also be noted that the above uniqueness result is obtained for the flexible-price model. For the sticky-price model matters are different. With sticky prices Benhabib, et al. [2] show that the resulting system is a saddlepath does not guarantee uniqueness since both variables (λ and π in their case) are jump variables. For the sticky-price model in Dupor [6], adding one initial condition for capital is still not sufficient to ensure uniqueness since there may be a second dimension of indeterminacy in the subsystem (λ, π) that leads to one-dimensional indeterminacy in the full (λ, π, k) system.

Before closing this section, we note that, as in Proposition 1, the uniqueness result also holds in the case of the general interest-rate feedback rule whereby the

[6] See Benhabib et al. [2] for a summary of the results for the model without capital.

nominal interest rate is set as an increasing function of both inflation rate and output. Specifically,

$$R_t = R(\pi_t, y_t) = \psi(\pi_t, k_t) \tag{18}$$

where $\psi(\pi_t, k_t)$ is non-decreasing in π and k ($\psi_\pi \geqslant 0$, $\psi_k \geqslant 0$), continuous, differentiable, and strictly positive. The monetary policy is referred to as active if $\psi_\pi > 1$ and passive if $\psi_\pi < 1$. The general interest-rate feedback rule in (18) is in line with the rule proposed by Taylor [14].[7]

With the general interest-rate rule (18) replacing the simple rule (9), Eq. (6) can be written as $f'(k_t) + \pi_t = \psi(\pi_t, k_t)$, from which π_t can be solved, i.e., $\pi_t = \pi(k_t)$. By substitution we can express c_t and m_t as functions of λ_t and k_t, and obtain the following differential equations

$$\dot{k}_t = f(k_t) - c(\lambda_t, k_t) \tag{19}$$

$$\dot{\lambda}_t = \lambda_t(\rho - f'(k_t)) \ . \tag{20}$$

Linearizing at the steady state (λ^*, k^*), we have

$$\begin{pmatrix} \dot{\lambda}_t \\ \dot{k}_t \end{pmatrix} = A_1 \begin{pmatrix} \lambda_t - \lambda^* \\ k_t - k^* \end{pmatrix} . \tag{21}$$

The determinant of the Jacobian matrix, $\det(A_1) = -c_\lambda(\lambda^*, k^*)\lambda^* f''(k^*) < 0$, which again implies that there is equilibrium uniqueness.

3 Productive money

In this section we allow for a role played by the demand for money by firms in the monetary transmission mechanism, as is emphasized by Benhabib et al. [2]. For this purpose we separate real money balances (m_t) into two parts, productive money (m_t^p) and nonproductive money (m_t^{np}), i.e., $m_t = m_t^p + m_t^{np}$. The household's lifetime utility is given by

$$U = \int_0^\infty e^{-\rho t} u(c_t, m_t^{np}) dt \ . \tag{22}$$

The household's budget constraint is now given by

$$c_t + \dot{m}_t + \dot{b}_t + \dot{k}_t = (R_t - \pi_t) b_t + y_t - \pi_t m_t - \tau_t \tag{23}$$

where the production technology is specified as follows:

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[7] Taylor [14] argues that from 1984 to 1992 the Fed followed a linear interest-rate feedback rule by setting the federal funds rate as $R_t = 1.5(\pi_t - \pi^*) + 0.5(y_t - y^*)$, where π^* and y^* are the target inflation rate and output respectively.

Assumption 2′. *The production function $y_t = f(k_t, m_t^p)$ is strictly increasing and strictly concave in its arguments, and the production inputs are complementary to each other, i.e.,*

$$f_1 > 0, f_2 > 0, f_{11} < 0, f_{22} < 0, f_{11}f_{22} - f_{12}^2 > 0, \text{ and } f_{12} > 0$$

where the numerical subscripts denote partial derivatives of the production function with respect to its ith argument. The complementarity assumption of capital and money in the production function, i.e., $f_{12} > 0$, implies that the more money the firm holds, the more productive the physical capital input. The budget constraint for this case is given by

$$\dot{a}_t = (R_t - \pi_t)a_t + f(k_t, m_t^p) - (R_t - \pi_t)k_t - R_t(m_t^{np} + m_t^p) - c_t - \tau_t \ . \tag{24}$$

The first-order conditions are

$$u_c(c_t, m_t^{np}) = \lambda_t \tag{25}$$

$$\frac{u_m(c_t, m_t^{np})}{u_c(c_t, m_t^{np})} = R(\pi_t) \tag{26}$$

$$f_1(k_t, m_t^p) = R(\pi_t) - \pi_t \tag{27}$$

$$f_2(k_t, m_t^p) = R(\pi_t) \tag{28}$$

$$\dot{\lambda}_t = \lambda_t(\rho + \pi_t - R(\pi_t)) \ . \tag{29}$$

Equations (27) and (28) can be used to solve π_t and m_t^p as functions of k_t, that is, $\pi_t = \pi(k_t)$, $m_t^p = m^p(k_t)$, and

$$\pi' \equiv d\pi/dk = -\frac{(f_{11}f_{22} - f_{12}^2)}{(1 - R')f_{22} + R'f_{12}} \ . \tag{30}$$

Similarly, from Eqs. (25) and (26), we have $c_t = c(\lambda_t, k_t)$, $m_t^{np} = m^{np}(\lambda_t, k_t)$. In addition

$$\partial c/\partial\lambda = (u_{mm} - Ru_{cm})/\Delta < 0 \ . \tag{31}$$

From the market-clearing condition and Eq. (29), we obtain the following dynamic system in terms of k_t and λ_t

$$\dot{k}_t = f(k_t, m^p(k_t)) - c(\lambda_t, k_t), \tag{32}$$

$$\dot{\lambda}_t = \lambda_t[\rho + \pi(k_t) - R(\pi(k_t))]. \tag{33}$$

By linearization we have

$$\begin{pmatrix} \dot{k}_t \\ \dot{\lambda}_t \end{pmatrix} = A_2 \begin{pmatrix} k_t - k^* \\ \lambda_t - \lambda^* \end{pmatrix} \tag{34}$$

where the determinant of the Jacobian matrix A_2 is

$$\det(A_2) = \lambda^* \pi'(1 - R')(\partial c / \partial \lambda) \,.$$

From (30), if $R'(\pi^*) > 1$, then $\pi' < 0$, which, together with (31), implies that $\det(A_2) < 0$ and the two eigenvalues have opposite signs. We therefore have the following result:

Proposition 2. *In the case with both capital and productive money, if monetary policy is active* ($R'(\pi^*) > 1$)*, then there exists a unique perfect foresight equilibrium in which all variables converge asymptotically to the steady state.*

The result in Proposition 2 is in contrast to that obtained in Benhabib et al. [2], in which with productive money and without endogenous investment active monetary policy is likely to lead to indeterminacy. Here by adding a predetermined variable, physical capital, and under Assumption $2'$, active monetary policy stabilizes the economy by bringing about real determinacy.

On the other hand, in the present case with productive money and capital, depending upon parameter values in the model there can be uniqueness or indeterminacy (or even no convergent equilibrium) if monetary policy is passive.

4 Endogenous labor supply

In this section we consider the extension that includes the endogenous labor-leisure choice. That is, the household's lifetime utility is given by

$$U = \int_0^{\infty} e^{-\rho t}[u(c_t, m_t, l_t)]dt \tag{35}$$

where l_t denotes the labor supply. We first study the case with separable leisure, and then examine the case with nonseparable leisure.

4.1 The separable leisure case

We show below that with separable leisure there is equilibrium uniqueness under both active and passive monetary policies.

The household's lifetime utility is

$$U = \int_0^{\infty} e^{-\rho t}[u(c_t, m_t) - v(l_t)]dt \tag{36}$$

where $v(\cdot)$ is strictly increasing and convex, i.e., $v'(\cdot) > 0$ and $v''(\cdot) \geqslant 0$. The production technology is modified as follows:

Assumption 2$''$. *The constant-returns-to-scale production function $y_t = f(k_t, l_t)$ is strictly increasing and strictly concave in its arguments, and capital and labor inputs are complementary to each other, i.e.,*

$$f_i\,(k_t, l_t) > 0 > \ f_{ii}(k_t, l_t), i = k, l, \text{ and } f_{kl} > 0\ . \tag{37}$$

The optimality conditions associated with the household-firm's problem are

$$u_c\,(c_t, m_t) = \lambda_t \tag{38}$$

$$\frac{u_m\,(c_t, m_t)}{u_c\,(c_t, m_t)} = R(\pi_t) \tag{39}$$

$$f_k(k_t, l_t) = R(\pi_t) - \pi_t \tag{40}$$

$$v'(l_t) = \lambda_t f_l(k_t, l_t) \tag{41}$$

$$\dot{\lambda}_t = \lambda_t\,(\rho + \pi_t - R(\pi_t)) \tag{42}$$

$$\lim_{t\to\infty} e^{-\int_0^\infty [R(s) - \pi(s)]ds} a_t = 0. \tag{43}$$

along with the market-clearing condition

$$\dot{k}_t = f(k_t, l_t) - c_t\ . \tag{44}$$

From Eqs. (40) and (41) and by the implicit function theorem, we have $l_t = l(\lambda_t, k_t)$, $\pi_t = \pi(\lambda_t, k_t)$, where

$$l_\lambda = \frac{f_l}{v'' - \lambda f_{ll}} > 0$$

$$l_k = \frac{\lambda f_{kl}}{v'' - \lambda f_{ll}} > 0$$

$$(1 - R')\pi_\lambda = -\frac{f_l f_{kl}}{(v'' - \lambda f_{ll})} < 0$$

$$(1 - R')\pi_k = -\frac{v'' f_{kk}}{(1 - R')(v'' - \lambda f_{ll})} > 0\ .$$

In addition, from Eqs. (38) and (39), we can solve for $c_t = c(\lambda_t, k_t)$. The system reduces to two differential equations

$$\dot{\lambda}_t = \lambda_t[\rho + \pi(\lambda_t, k_t) - R(\pi(\lambda_t, k_t))] \tag{45}$$

$$\dot{k}_t = f(k_t, l(\lambda_t, k_t)) - c(\lambda_t, k_t)\ . \tag{46}$$

Evaluated at the steady state the Jacobian matrix for the linearized system is given by W

$$A_3 = \begin{bmatrix} \lambda^*(1-R')\pi_\lambda & \lambda^*(1-R')\pi_k \\ f_l l_\lambda - c_\lambda & f_k + f_l l_k - c_k \end{bmatrix}$$

whose determinant is

$$\det(A_3) = \lambda^*[(1-R')\pi_\lambda(f_k + f_l l_k) - (1-R')\pi_k f_l l_\lambda] + \frac{\lambda^*[(1-R')\pi_k](u_{mm} - \frac{u_m}{u_c}u_{cm})}{(u_{cc}u_{mm} - u_{cm}^2)} < 0\ .$$

We thus have the following proposition:

Proposition 3. *With nonseparable leisure, under both active and passive monetary policies (i.e., for both $R'(\pi^*) > 1$ and $R'(\pi^*) < 1$), there exists a unique perfect foresight equilibrium in which all variables converge asymptotically to the steady state.*

4.2 The non-separable leisure case

We now consider the preference specification with nonseparable leisure. It is important to investigate this case, because, intuitively, if leisure is not separable, then monetary policy can have significant effects on production through its influence on labor supply. We show below this may indeed be the case, and that equilibrium indeterminacy can arise under both passive and active monetary policies. The results below also suggest that applied researchers in this area need to know whether certain of their numerical methods are valid when nonseparable leisure is specified in quantitative business cycle models.[8]

One class of utility functions with nonseparable leisure suggested by Farmer [7] is, $\frac{X(c,m)^{1-\sigma}}{1-\sigma} - W(c,m)^{1-\sigma} l^{1+\chi}$, where $\sigma > 1$, $\chi \geq 0$ and both $X(\cdot)$ and $W(\cdot)$ take the CES forms. The assumption of the CES forms of the period utility functions, $X(\cdot)$ and $W(\cdot)$, is made to admit the existence of a well-defined steady state.[9] Farmer also considers a special case of this class of utility functions, i.e.,

$$\frac{c_t^{1-\sigma}}{1-\sigma} - m_t^{1-\sigma} l_t^{1+\chi} \tag{47}$$

[8] Many solution methods require determinacy.

[9] See Farmer [7] for more details.

and shows that under this preference specification, equilibrium indeterminacy can arise under interest-rate pegging (i.e., the government sets nominal interest rate at a fixed level). Following Farmer, below we adopt the preference specification given in (47). Note that since interest-rate pegging is a special case of passive interest-rate policies (equivalent to the case $R' = 0 < 1$), the analysis here can be considered as an extension of Farmer's work to interest-rate feedback policies.

In addition we adopt the Cobb-Douglas production function, i.e.,

$$f(k_t, l_t) = l_t^{\alpha} k_t^{1-\alpha} \; . \tag{48}$$

Obviously, the Cobb-Douglas function satisfies all the properties in Assumption $2''$ given in the previous subsection.

The optimality conditions are given by

$$c_t^{-\sigma} = \lambda_t \tag{49}$$

$$(\sigma - 1) m_t^{-\sigma} l_t^{1+\chi} = \lambda_t R(\pi_t) \tag{50}$$

$$(1-\alpha) l_t^{\alpha} k_t^{-\alpha} = R(\pi_t) - \pi_t \tag{51}$$

$$(1+\chi) m_t^{1-\sigma} l_t^{\chi} = \lambda_t \alpha l_t^{\alpha-1} k_t^{1-\alpha} \tag{52}$$

$$\dot{\lambda}_t = \lambda_t \left(\rho + \pi_t - R(\pi_t)\right) \tag{53}$$

together with the market-clearing condition

$$\dot{k}_t = l_t^{\alpha} k_t^{1-\alpha} - c_t \; .$$

From the Appendix, the dynamic system boils down to the following two differential equations

$$\begin{pmatrix} \dot{\lambda}_t \\ \dot{k}_t \end{pmatrix} = A_4 \begin{pmatrix} \lambda_t - \lambda^* \\ k_t - k^* \end{pmatrix} \tag{54}$$

where

$$tr(A_4) = \frac{1}{\Omega} \left[(1+\chi-\alpha)\left(1 - R'(\pi^*)\right) + \frac{(1-\sigma)\rho}{1-\alpha} \frac{\alpha R'(\pi^*)}{\rho+\pi^*} \right] \tag{55}$$

$$\det(A_4) = \frac{\alpha\rho(1+\chi)(R'(\pi^*) - 1)}{(1-\alpha)\sigma\Omega} \tag{56}$$

and $\Omega \equiv \left[(1+\chi-\alpha\sigma)\left(\frac{1-R'(\pi^*)}{\rho}\right) + (1-\sigma)\frac{\alpha R'(\pi^*)}{\rho+\pi^*} \right]$.

Note that for $R'(\pi^*) > 1$ and parameter values $1+\chi-\alpha\sigma > 0$, $\det(A_4) < 0$ and there is equilibrium uniqueness. On the other hand for $R'(\pi^*) > 1$ and parameter values $1+\chi-\alpha\sigma < 0$, the equilibrium dynamics depends on the magnitude of π^*. In particular, if $\pi^* = 0$, then $\det(A_4) < 0$, and we have equilibrium uniqueness. In addition, for given value of $R'(\pi^*)$ and sufficiently large π^*, $\det(A_4) > 0$ and $tr(A_4) < 0$, equilibrium indeterminacy will occur. Thus we have the following results.

Proposition 4. *For the preference with nonseparable leisure given in (47) and active monetary policy ($R'(\pi^*) > 1$),*

i) if $1 + \chi - \alpha\sigma > 0$, there is equilibrium uniqueness.
ii) if $1 + \chi - \alpha\sigma < 0$, there is equilibrium uniqueness for sufficiently low steady-state inflation rate, i.e., $\pi^* < \bar{\pi}$, and equilibrium indeterminacy for sufficiently high steady-state inflation rate, i.e., $\pi^* > \bar{\pi}$, where $\bar{\pi} = \rho[(\frac{\sigma-1}{1-\chi-\alpha\sigma})(\frac{\alpha R'}{1-R'}) - 1]$.

Examples for case ii) in the above proposition can be constructed easily. For instance, given parameter values $\chi = 0$, $\alpha = 0.66$, $\rho = 0.0045$ and $\sigma = 1.6$, then $\bar{\pi} \approx 9.10\%$ when $R'(\pi^*) = 1.6$, and $\bar{\pi} \approx 3.69\%$ when $R'(\pi^*) = 1.7$.[10] Note that the condition, $1+\chi-\alpha\sigma < 0$, plays an important role for the indeterminacy result. To see this indeterminacy condition more clearly, we rewrite it as

$$\sigma \left[(\frac{1+\chi}{\sigma} - 1) - (\alpha - 1) \right] < 0 \, .$$

The first term inside the square brackets is the slope of the reduced form labor supply curve while the second term represents the slope of the labor demand curve. As is pointed out by Farmer [7], this condition carries the same intuition as the real model in Benhabib and Farmer [1], i.e., the labor demand and supply curves cross with *wrong* slopes.[11]

Similarly, we can obtain the following proposition for the case of passive monetary policy.

Proposition 5. *For the preference with nonseparable leisure given in (47) and passive monetary policy ($R'(\pi^*) < 1$),*

i) if $1 + \chi - \alpha\sigma > 0$, indeterminacy occurs if $\pi_1 < \pi^* < \pi_2$, where $\pi_1 = \rho[(\frac{\sigma-1}{1+\chi-\alpha})(\frac{\alpha}{1-\alpha})(\frac{R'}{1-R'}) - 1]$, $\pi_2 = \rho[(\frac{\sigma-1}{1+\chi-\alpha\sigma})(\frac{\alpha R'}{1-R'}) - 1]$, and $\alpha + \sigma > 2 + \chi$.[12]
ii) if $1 + \chi - \alpha\sigma < 0$, there exists no equilibrium for sufficiently low steady-state inflation rate, i.e., $\pi^* < \hat{\pi}$, and equilibrium indeterminacy for sufficiently high steady-state inflation rate, i. e., $\pi^* > \hat{\pi}$, where $\hat{\pi} = \rho[(\frac{\sigma-1}{1+\chi-\alpha})(\frac{\alpha}{1-\alpha})(\frac{R'}{1-R'}) - 1]$.

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[10] The dependence of the indeterminacy condition on the value of the steady-state inflation rate in Proposition 4 is similar to the results obtained in Meng [10] in a cash-in-advance model with nonseparable leisure and without endogenous investment.

[11] Notice that there is a minor difference between our (or Farmer's) model and the real model in Benhabib and Farmer [1]. In the former the wrong slope comes from the downward labor supply curve, while in the latter it comes from the upward sloping labor demand curve.

[12] The condition $\alpha + \sigma > 2 + \chi$ ensures that $\pi_1 < \pi_2$ holds. Note also that although $\bar{\pi}$ and π_2 have the same expression, they correspond to different interest-rate policies and thus have different values. The same is true for π_1 and $\hat{\pi}$.

Examples can be given for both cases in Proposition 5 as in Proposition 4 for active monetary policy. Under the special passive interest-rate policy, i.e., interest-rate pegging ($R' = 0$), it is easy to see that $\det(A_4) > 0$ and $tr(A_4) < 0$ when $1 + \chi - \alpha\sigma < 0$, thus indeterminacy happens regardless of the magnitude of the steady-state inflation rate. This is the result obtained in a discrete-time model in Farmer [7]. However, unlike the general passive interest-rate policy considered here, indeterminacy can not occur under interest-rate pegging if $1 + \chi - \alpha\sigma > 0$. In the present case, indeterminacy can arise under passive monetary policy when the labor demand and supply curves cross with right slopes.

5 Conclusion

By incorporating neoclassical investment in the flexible-price model, this paper examines the role that physical capital plays in stabilizing the real side of the economy by ensuring uniqueness of equilibrium in alternative environments if the monetary authority follows interest-rate feedback rules. Our results demonstrate that physical capital as a sluggish variable with a given initial condition indeed is very important in affecting the outcomes. In particular, with inelastic labor supply equilibrium uniqueness is ensured under both active and passive monetary policies. For the case where money affects both preferences and technology, the uniqueness result remains true under active monetar policy. With endogenous labor supply, the uniqueness result holds again regardless of the stance of monetary policies for the case with separable leisure, but indeterminacy remains likely under both active and passive monetary policies when leisure is nonseparable.

We would like to point out that even the uniqueness results in this paper should be interpreted with caution. Following the logic in Benhabib et al. [3], if the feedback rules respect the zero lower bound on nominal rates global indeterminacy may arise. For example, consider the case with inelastic labor supply. If there exists a steady state with active monetary policy, there must exist another steady state in which monetary policy is passive. As a result, the economy can be placed on either of the two saddlepaths corresponding to the two locally determinate steady states given the initial condition k_0.[13]

Another possible extension is to investigate more general preference specifications with nonseparable leisure as in Farmer [7]. As is noted by Farmer, the special preference specification that we adopt in Section 4.2 is not rich enough to fit all the features of the data.[14] Such analysis may shed further light on how alternative

[13] In some cases, even more complicated results may arise. For instance, in the case with nonseparable leisure, it is possible that locally there exists a saddlepath for the steady state with active monetary policy, but indeterminacy for the steady state with passive monetary policy.

[14] Another candidate for preferences with nonseparable leisure (useful for applied work and quantitative analysis) can be found in Chari, Kehoe and McGrattan [4], i.e.,

$$u(c, l, m) = \frac{\left[\left(\omega c^{(\eta-1)/\eta} + (1-\omega) m^{(\eta-1)/\eta}\right)^{\eta/(\eta-1)} (1-l)^{\psi}\right]^{1-\sigma}}{1-\sigma}.$$

interest-rate feedback rules affect production and aggregate stability through their interaction with labor supply.

6 Appendix

In this appendix we derive the expressions (55) and (56) in the text. From Eq. (49), we have, $c_t = c(\lambda_t)$, and $c'(\lambda) = -c_t^{1+\sigma}/\sigma$. In addition, from Eqs. (50)–(52) we have $l_t = l(\lambda_t, k_t)$, and $\pi_t = \pi(\lambda_t, k_t)$, and at the steady state

$$\partial l/\partial\lambda = \frac{l^*}{\lambda^* \Omega}\left[\frac{1-R'(\pi^*)}{\rho}\right]$$

$$\partial l/\partial k = \frac{l^*}{k^* \Omega}\left[\sigma(1-\alpha)\left(\frac{1-R'(\pi^*)}{\rho}\right) + \frac{(1-\sigma)\alpha R'(\pi^*)}{\rho+\pi^*}\right]$$

$$\partial \pi/\partial\lambda = -\frac{\alpha}{\lambda^* \Omega}$$

$$\partial \pi/\partial k = \frac{\alpha(1+\chi-\sigma)}{k^* \Omega}$$

where $\Omega \equiv \left[(1+\chi-\alpha\sigma)\left(\frac{1-R'(\pi^*)}{\rho}\right) + (1-\sigma)\frac{\alpha R'(\pi^*)}{\rho+\pi^*}\right]$. The dynamic system is then given by

$$\begin{pmatrix}\dot{\lambda}_t \\ \dot{k}_t\end{pmatrix} = A_4\begin{pmatrix}\lambda_t - \lambda^* \\ k_t - k^*\end{pmatrix}$$

where

$$A_4 = \begin{bmatrix} \lambda^*(1-R')(\partial\pi/\partial\lambda) & \lambda^*(1-R')(\partial\pi/\partial k) \\ \alpha(l^*/k^*)^{\alpha-1}(\partial l/\partial\lambda) - (\partial c/\partial\lambda) & (1-\alpha)(l^*/k^*)^{\alpha} + \alpha(l^*/k^*)^{\alpha-1}(\partial l/\partial k)\end{bmatrix}$$

We can thus obtain the expressions for the trace and determinant of the above Jacobian matrix, i.e., (55) and (56) in the text.

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