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Exposita Notes

Indeterminacy in a small open economy with endogenous labor supply⋆

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Summary. We establish conditions under which indeterminacy can occur in a small open economy business cycle model with endogenous labor supply. Indeterminacy requires small externalities in technologies with social constant returns to scale, independently of the intertemporal elasticities in both consumption and labor.

Keywords and Phrases: Indeterminacy, Small open economy, Business cycles.

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1 Introduction

It is well understood by now that under some conditions closed-economy models of the business cycles can be subject to indeterminacy, in the sense that there exist a continuum of equilibrium trajectories converging to a steady state.[1] The literature on indeterminacy underscores that equilibria need not be uniquely determined by the fundamentals of the economy, and that the existence of indeterminate equilibria is associated with the possibility of self-fulfilling prophecies. The remaining question is how plausible the requirements are to generate such indeterminacy.

Early models relied on relatively large increasing returns to scale to generate indeterminacy.[2] The estimates by Hall ([10], [11]) and others made this plausible.

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[1] For an excellent survey of the literature, see Benhabib and Farmer [4].

[2] See, e.g., Benhabib and Farmer [2].

But over time, the empirical evidence has mounted against large increasing returns.[3] More recently, Benhabib and Farmer [3] showed that in a two-sector model the size of increasing returns need not be large. Benhabib and Nishimura [5] went further, showing that decreasing marginal costs are not necessary to render the steady state indeterminate. In their model, small production externalities with social constant returns are sufficient to generate multiple equilibria.

But these closed-economy models also place restrictions on preferences. They work best, in the sense of requiring only small market distortions to generate indeterminacy, either when the intertemporal elasticity of substitution in consumption or/and the elasticity on labor supply is high. The intuition for why these conditions are needed is straightforward. Take for instance, the two-sector model (with a consumption good sector and an investment good sector) of Benhabib and Farmer [3]. They provide the intuition for indeterminacy succinctly ([3], pp. 423):

> "Consider starting with an arbitrary equilibrium trajectory of investment or consumption, and inquire whether a faster rate of accumulation and growth can also be justified as an equilibrium. This would require a higher return on investment. If higher anticipated stocks of future capital raise the marginal product of capital by drawing labor out of leisure, or by reallocating labor across sectors, the expected higher rate of return may be self-fulfilling ... If . . . there are sufficient increasing returns that are consistent with optimization, either because of externalities or because of imperfect competition that generate markups, these increasing returns may amplify the movement of labor into production and provide a sufficient boost to private rates of returns to justify multiple equilibria. The critical parameters are the magnitudes of increasing returns or externalities, and the ease with which labor can be drawn into employment – that is – the elasticity of labor supply."

Benhabib and Farmer [3] assume a utility function that is logarithmic in consumption. With that formulation it is relatively costly to reallocate labor out of the consumption good sector, and high labor elasticity is needed in order to draw labor out of leisure and to raise the marginal product of capital. If, on the other hand, one assumes larger values of intertemporal elasticity of substitution in consumption, then high labor elasticity is not required for indeterminacy. In particular, if one uses linear consumption (i.e., the intertemporal elasticity in consumption is infinite), then indeterminacy can arise in a broader range of values for labor elasticity. Obviously, with linear consumption labor can be freely reallocated from the consumption good sector (in order to increase the marginal product of capital) with little effect on leisure. In that case, indeterminacy can arise even when labor supply is fixed.[4]

In this paper we extend research on indeterminacy to a small open economy model of the business cycle, in a way that addresses some of the limitations of earlier

[3] On recent empirical estimates, see Basu and Fernald [1] and Burnside, Eichenbaum and Rebelo [7]. These papers find little evidence of increasing marginal returns. Indeed, they conclude that returns to scale are roughly constant and that market imperfections are small.

[4] The same intuition applies to the model in Benhabib and Nisimura [5], and indeed it may be more pronounced there. With linear consumption, they show that indeterminacy conditions are in fact independent of labor elasticity.

theorizing. We combine the preferences proposed by Greenwood, Hercowitz and Huffman [9] with the technologies of social constant returns to scale introduced by Benhabib and Nishimura [5]. In that setup, indeterminacy can occur regardless of the elasticities on both consumption and labor, for technologies with very small or even negligible external effects. Thus, for open economies facing a perfect bond market, indeterminacy can obtain under empirically plausible conditions.

This paper is also a realistic extension of Weder [15] and related work in the literature, in that we incorporate endogenous labor supply to an otherwise standard Ramsey model of a small open economy. In such a small open economy model with inelastic labor supply, Weder [15] shows that indeterminacy can obtain more easily than a closed-economy variant of Benhabib and Farmer [3].[5] But with fixed labor supply, employment (or unemployment) fluctuations – a key element in business cycle fluctuations – simply can not be explained. In this paper we bring this feature back into the picture. An additional advantage of our analysis is that we can derive an explicit closed-form condition for indeterminacy, something that is not possible in models with endogenous labor supply and other parameter specifications.[6]

2 The two-sector open economy with endogenous labor supply

Consider a small open economy inhabited by an infinite-lived representative agent who maximizes the intertemporal utility function

$$\int_0^\infty u(c_t, l_t)e^{-\rho t}dt \tag{1}$$

where c_t is the consumption of traded goods, l_t labor supply and ρ the parameter of time preference. Assume the economy is open to full international capital mobility, so that the domestic representative agent can borrow from and lend to the outside world freely. This agent has access to net foreign bonds d_t, denominated in units of the tradable good, that pay a world interest rate r, which is exogenously given to the small open economy. Assume that consumption goods are tradable and capital goods non-tradeable, as in Weder [15].

On the production side there are two sectors: one producing the consumption good (y_{1t}) and the other producing the investment good (y_{2t}). Following Benhabib and Nishimura [5], the production functions are assumed to be Cobb-Douglas with externality components

$$y_{1t} = A_1 l_{1t}^{\alpha_0} k_{1t}^{\alpha_1} \tag{2}$$

$$y_{2t} = A_2 l_{2t}^{\beta_0} k_{2t}^{\beta_1} \tag{3}$$

[5] With inelastic labor supply, Meng and Velasco [14] obtain similar results in an open economy version of Benhabib and Nishimura [5]. In a small open economy endogenous growth model, again with fixed labor supply, Lahiri [12] shows also that it is easier to obtain multiple transitional growth paths than in the closed economy.

[6] Therefore, with endogenous labor supply there are situations under which indeterminacy can occur in the closed economy, but it is unclear whether indeterminacy can also happen in the open economy.

where $A_1 = \overline{l_{1t}^{a_0} k_{1t}^{a_1}}$, $A_2 = \overline{l_{2t}^{b_0} k_{2t}^{b_1}}$, $\sum_{i=0}^{1} (\alpha_i + a_i) = \sum_{i=0}^{1} (\beta_i + b_i) = 1$, and

$$l_{1t} + l_{2t} = l_t, \quad k_{1t} + k_{2t} = k_t \tag{4}$$

Here l_{1t} and k_{1t} denote the capital and labor services used by the individual firm in the consumption goods producing sector, and l_{2t} and k_{2t} for the investment goods producing sector. The components A_1 and A_2 of the production functions represent external effects that are viewed as functions of time by the agent.

Constant returns coupled with small external effects imply that some sectors must display a small degree of decreasing returns at the private level. This is in contrast to models of indeterminacy with social increasing, but private constant returns to scale.[7]

The agent's budget constraint is

$$\dot{d}_t = rd_t + y_{1t} + p_t y_{2t} - c_t - p_t i_t \tag{5}$$

where p_t is the relative price of the nontradeable investment good to the tradeable consumption good. Sometimes this price is referred to as the real exchange rate. Note that in (5) the consumption good is taken to be the numeraire. Note also that p_t is taken as exogenously given by the agent, but is determined by market-clearing conditions. The variable i_t denotes gross investment, so that the law of motion for capital is

$$\dot{k}_t = i_t - \delta k_t \tag{6}$$

Equations (5) and (6) can be consolidated into

$$\dot{a}_t = ra_t + y_{1t} + p_t y_{2t} - c_t + k_t(\dot{p}_t - rp_t - \delta p_t) \tag{7}$$

where $a_t = d_t + p_t k_t$. The agent's problem is to choose $c_t, l_{1t}, l_{2t}, i_t, k_{1t}, k_{2t}$ and d_t to maximize (1), subject to (2), (3), (4) and (7), and given k_0 and d_0.

The Hamiltonian is

$$\begin{aligned}\mathcal{H} = {} & u(c_t, l_t) + \lambda_t(ra_t + y_{1t} + p_t y_{2t} - c_t + k_t(\dot{p}_t - rp_t - \delta p_t)) + \\ & \mu_t(k - k_{1t} - k_{2t}) + \omega_t(l_t - l_{1t} - l_{2t})\end{aligned}$$

where λ_t is a costate; μ_t and ω_t are the rental rate of the investment good and the wage rate of labor, all in terms of the consumption good. The first-order conditions are

$$u_c(c_t, l_t) = \lambda_t \tag{8}$$

$$u_l(c_t, l_t) = \beta_0 \lambda_t p_t l_{2t}^{\beta_0 + b_0 - 1} k_{2t}^{\beta_1 + b_1} \tag{9}$$

$$\omega_t = \lambda_t \alpha_0 l_{1t}^{\alpha_0 + a_0 - 1} k_{1t}^{\alpha_1 + a_1} = \lambda_t \beta_0 p_t l_{2t}^{\beta_0 + b_0 - 1} k_{2t}^{\beta_1 + b_1} \tag{10}$$

[7] Although we adopt the production functions with social constant returns to scale, similar results to those obtained below carry over to the case of increasing returns to scale that is specified in Benhabib and Farmer [3] and used in Weder [15]. We use the Benhabib and Nishimura [5] setup for expositional simplicity.

$$\mu_t = \lambda_t \alpha_1 l_{1t}^{\alpha_0+a_0} k_{1t}^{\alpha_1+a_1-1} = \lambda_t \beta_1 p_t l_{2t}^{\beta_0+b_0} k_{2t}^{\beta_1+b_1-1} \tag{11}$$

$$\dot{\lambda}_t = \lambda_t(\rho - r) \tag{12}$$

$$\dot{p}_t = p_t(r + \delta - \beta_1 l_{2t}^{\beta_0+b_0} k_{2t}^{\beta_1+b_1-1}), \tag{13}$$

together with the transversality conditions

$$\lim_{t\to\infty} \lambda_t d_t e^{-\rho t} = \lim_{t\to\infty} \lambda_t p_t k_t e^{-\rho t} = 0. \tag{14}$$

Preference structure

We adopt the following utility function popularized by Greenwood, Hercowitz and Huffman [9]:

$$\frac{1}{1-\sigma}[(c_t - \frac{1}{1+\chi} l_t^{1+\chi})^{1-\sigma} - 1] \tag{15}$$

where $\sigma > 0$ is the inverse of the intertemporal elasticity of substitution in consumption, and χ corresponds to the intertemporal elasticity of substitution in labor supply. The utility function form in (15) implies that the marginal rate of substitution between consumption and labor effort depends on the latter only[8]

$$-u_l(c_t, l_t)/u_c(c_t, l_t) = l_t^{\chi}, \tag{16}$$

so that labor effort is determined independently of the intertemporal consumption-savings choice. Such a property is essential in obtaining the indeterminacy results below.[9]

As is standard in international macroeconomics, we impose $\rho = r$, a condition that ensures a well-defined steady-state with constant bond-holdings. This assumption will also imply, by (12), that marginal utility remains constant over all time –that is, $\lambda_t = \bar{\lambda}$. Substituting $\lambda_t = \bar{\lambda}$ into other first-order conditions, by (8), (9) and (16) we have

$$c_t - \frac{1}{1+\chi} l_t^{1+\chi} = \bar{\lambda}^{-\frac{1}{\sigma}} \tag{17}$$

$$l_t^{\chi} = \beta_0 p_t l_{2t}^{\beta_0+b_0-1} k_{2t}^{\beta_1+b_1} \tag{18}$$

Note from (17) that when marginal utility remains at a constant level, so does the instant utility (15). Dividing (11) by (10) yields

$$\frac{\alpha_1}{\alpha_0} \frac{l_{1t}}{k_{1t}} = \frac{\beta_1}{\beta_0} \frac{l_{2t}}{k_{2t}} \tag{19}$$

Using (11) and (19) to solve for $\frac{l_{2t}}{k_{2t}}$, we have

$$\frac{l_{2t}}{k_{2t}} = \eta p_t^{\frac{1}{\alpha_0+a_0-\beta_0-b_0}} = \eta p_t^{\frac{1}{(\alpha_0+a_0)(\beta_1+b_1)-(\alpha_1+a_1)(\beta_0+b_0)}} \equiv g(p_t) \tag{20}$$

[8] We ignore the extreme case when $\chi = 0$, which implies that consumption and leisure are perfect substitutes.

[9] While Greenwood, Hercowitz and Huffman [9] first proposed a utility function of the form in (15) for the closed economy, it was later used for small open economy real business cycle models by a number of authors. See, e.g., Mendoza [13] and Correia et al. [8].

where $\eta = \frac{\beta_1}{\alpha_1}\left(\frac{\alpha_1\beta_0}{\alpha_0\beta_1}\right)^{\alpha_0+a_0}$. Thus, from (18) we have

$$l_t = \left[\beta_0 p_t g(p_t)^{\beta_0+b_0-1}\right]^{\frac{1}{\chi}} = l(p_t) \tag{21}$$

which is the equilibrium labor supply equation. Substituting (4), (19) and (20) into (21), we can solve for k_{2t}

$$k_{2t} = \frac{\alpha_0\beta_1}{\alpha_0\beta_1 - \alpha_1\beta_0} k_t + h(p_t) \tag{22}$$

where

$$h(p_t) = -\frac{\alpha_1\beta_0}{\alpha_0\beta_1 - \alpha_1\beta_0}\frac{\left[\beta_0 p_t g(p_t)^{\beta_0+b_0-1}\right]^{\frac{1}{\chi}}}{g(p_t)} \tag{23}$$

In addition, the market clearing conditions for the investment good and the economy's current account are, respectively

$$\dot{k}_t = l_{2t}^{\beta_0+b_0} k_{2t}^{\beta_1+b_1} - \delta k_t \tag{24}$$

$$\dot{d}_t = r d_t + l_{1t}^{\alpha_0+a_0} k_{1t}^{\alpha_1+a_1} - c_t \tag{25}$$

Substituting (20) and (22) into (13) and (24), we obtain the following differential equations for k_t and p_t

$$\dot{p}_t = p_t\left[r + \delta - \beta_1 g(p_t)^{\beta_0+b_0}\right] \tag{26}$$

$$\dot{k}_t = \left[\frac{\alpha_0\beta_1}{\alpha_0\beta_1 - \alpha_1\beta_0} g(p_t)^{\beta_0+b_0} - \delta\right] k_t + h(p_t) g(p_t)^{\beta_0+b_0} \tag{27}$$

These two equations describe the dynamics of the economy. The solution to this system can then be used, in conjunction with the other conditions laid out above, to solve for all variables of interest. In particular, the current account equation (25) and the transversality (14) combined determine the equilibrium consumption profile. To see this, integrate over (25), by using (17) and (21), to obtain

$$\int_0^\infty \left(de^{-rt}\right)' dt = \int_0^\infty y_{1t}(p_t, k_t) e^{-rt} dt - \int_0^\infty \left(\frac{1}{1+\chi} l_t^{1+\chi}(p_t) + \bar{\lambda}^{-\frac{1}{\sigma}}\right) e^{-rt} dt \tag{28}$$

Using the transversality condition we have

$$\bar{\lambda} = \left[r\int_0^\infty (y_{1t}(p_t, k_t) - \frac{1}{1+\chi} l_t^{1+\chi}(p_t)) e^{-rt} dt - r d_0\right]^{-\sigma} \tag{29}$$

Consumption can be obtained from ((17), i.e.

$$c_t = \frac{1}{1+\chi} l_t^{1+\chi}(p_t) + \bar{\lambda}^{-\frac{1}{\sigma}} \tag{30}$$

Therefore, once the solution path for (k_t, p_t) is determined, other variables including consumption can all be uniquely determined, and at the same time the transversality condition is satisfied. That is, indeterminacy in the two-by-two system in k_t and p_t implies that the overall economy is indeterminate.

3 The indeterminacy result

Linearizing around the steady state (p^*, k^*), the dynamics of k_t and p_t can be approximated by

$$\begin{pmatrix} \dot{p}_t \\ \dot{k}_t \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} p_t - p^* \\ k_t - k^* \end{pmatrix} \tag{31}$$

where

$$a_{11} = \frac{\beta_1(\beta_0 + b_0)p^* g(p^*)^{\beta_0 + b_0 - 1}}{(\alpha_1 + a_1)(\beta_0 + b_0) - (\alpha_0 + a_0)(\beta_1 + b_1)} \tag{32}$$

$$a_{22} = \frac{\alpha_0 r + \alpha_0 \delta(1 - \beta_1) + \alpha_1 \beta_0}{\alpha_0 \beta_1 - \alpha_1 \beta_0} \tag{33}$$

The eigenvalues of the Jacobian matrix in (31) are a_{11} and a_{22}. Notice that k_t is a predetermined variable and evolves continuously. p_t is a non-predetermined variable, that is, its initial value is not given by history and it may jump instantaneously in response to new information. Two negative eigenvalues mean equilibrium indeterminacy. We therefore have the following proposition:

Proposition. *In the two-sector open economy with endogenous labor supply, if the investment goods producing sector is labor intensive from the private perspective* $(\alpha_0\beta_1 - \alpha_1\beta_0 < 0)$ *and capital intensive from the social perspective* $((\alpha_1 + a_1)(\beta_0 + b_0) - (\alpha_0 + a_0)(\beta_1 + b_1) < 0)$*, then there exist a continuum of trajectories that converge to the steady state, i.e., the steady state is indeterminate.*

It is obvious that examples satisfying the indeterminacy conditions in the proposition can be constructed with arbitrarily small external effects. The indeterminacy result implies that the first order conditions and the transversality conditions are not sufficient to determine a unique solution path. Essentially, multiplicity of this sort indicates that the rational expectations equilibria involve random variables that are unrelated to the economy's fundamentals simply because agents believe it to be so.

4 Discussion and extensions

The conditions for indeterminacy in the above proposition are similar to those in the closed economy when the utility function is linear in consumption. Without externalities, the two eigenvalues are of opposite signs due to the Stolper-Samuelson and Rybczynski effects. With small externalities, however, the link between these two effects may be broken so that indeterminacy can occur.

It is important to note that the above indeterminacy conditions are independent of the preference parameters. Two key factors make this possible. In this open economy facing a perfect world bond market, if the agent wants to invest more to jump onto a new equilibrium trajectory she does not need to curtail her consumption. And for the assumed preference specification, labor supply is independent of intertemporal substitution effects (i.e., λ_t), and so is the decision on the capital goods production.

Both of the two factors are indispensable for the indeterminacy result. If we instead used the following utility function,

$$u(c_t, l_t) = (1-\sigma)^{-1}(c_t^{1-\sigma} - 1) - (1+\chi)^{-1} l_t^{1+\chi}, \tag{34}$$

we would not be able to give a definitive answer as to if and under what conditions indeterminacy can occur. To see this, substitute (34) into (9)

$$l_t^{\chi} = \beta_0 \bar{\lambda} p_t l_{2t}^{\beta_0+b_0-1} k_{2t}^{\beta_1+b_1} \tag{35}$$

In this case, labor supply is not independent of marginal utility, so that equation (21) must be modified to include $\bar{\lambda}$, which in turn implies that the dynamic equation (27) depends upon $\bar{\lambda}$ as well. Therefore one cannot solve recursively for the two-by-two system of k_t and p_t first, and then use the current account equation (along with the transversality condition) to obtain $\bar{\lambda}$ and c_t as we do here. Alternatively, one can view the dynamic problem consisting of four differential equations for k_t, p_t, λ_t and d_t in which there exists a zero eigenvalue. Unlike the previous case with utility function (15), closed form solutions for $\bar{\lambda}$ and c_t cannot be obtained. Thus, one would not be able to derive an indeterminacy condition in this case, even if such a condition existed at all. The same is true of other types of standard utility functions used in the real business cycle literature, e.g., $u(c_t, l_t) = \frac{1}{1-\sigma}[(c_t^{\theta}(1-l_t)^{1-\theta})^{1-\sigma} - 1]$.[10]

Finally, if the utility function (15) were applied to the closed economy in Benhabib and Nishimura [5], then indeterminacy could happen for low labor elasticity. But the trade-off would be that σ must be low as well. For instance, if $\sigma = 0$, so that utility function is linear in consumption, indeterminacy could occur for any degree of labor elasticity, a result already obtained in Benhabib and Nishimura [5]. In fact, the two utility functions in (15) and (34) are equivalent in this sense if and only if $\sigma = 0$.

References

1. Basu, S., Fernald, J. G.: Returns to scale in U.S. production: estimates and implications. Journal of Political Economy **105**, 249–283 (1997)
2. Benhabib, J., Farmer, R. E.: Indeterminacy and increasing returns. Journal of Economic Theory **63**, 19–41 (1994)
3. Benhabib, J., Farmer, R. E.: Indeterminacy and sector-specific externalities. Journal of Monetary Economics **37**, 397–419 (1996)
4. Benhabib, J., Farmer, R. E.: Indeterminacy and sunspots in macroeconomics. In: Taylor, J. B., Woodford, M. (eds.) Handbook of macroeconomics, Vol. 1A, pp. 387–448. New York: North-Holland 1999
5. Benhabib, J., Nishimura, K.: Indeterminacy and sunspots with constant returns. Journal of Economic Theory **81**, 58–96 (1998)
6. Bennett R., Farmer, R. E.: Indeterminacy with non-separable utility. Journal of Economic Theory **93**, 118–143 (2000)
7. Burnside, C., Eichenbaum, M., Rebelo, S.: Capital utilization and returns to scale. NBER Macroeconomics Annual **10**, 67–110 (1995)

[10] For the same reason, the non-separable utility function introduced for the closed economy in Bennett and Farmer [6], $u(c_t, l_t) = [(c_t v(l_t))^{1-\sigma} - 1]/(1-\sigma)$, cannot be applied to the open economy either.

8. Correia, I., Neves, J., Rebelo, S.: Business cycles in a small open economy. European Economic Review **39**, 1089–1113 (1995)
9. Greenwood, J., Hercowitz, Z., Huffman G.: Investment, capacity utilization and the real business cycle. American Economic Review **78**, 402–417 (1988)
10. Hall, R. E.: The relation between price and marginal cost in the U.S. industry. Journal of Political Economy **96**, 921–948 (1988)
11. Hall, R. E.: Invariance properties of Solow's productivity residual. In: Diamond, P. (ed.) Growth, productivity, unemployment, pp. 71–112. Cambridge, MA: MIT Press 1990
12. Lahiri, A.: Growth and equilibrium indeterminacy: the role of capital mobility. Economic Theory **17**, 197–208 (2001)
13. Mendoza, E.: Real business cycles in a small open economy. American Economic Review **81**, 797–818 (1991)
14. Meng, Q., Velasco, A.: Indeterminacy under constant returns to scale in open economies. Mimeo, Chinese University of Hong Kong (2000)
15. Weder, M.: Indeterminacy in the small open economy Ramsey growth model. Journal of Economic Theory **98**, 339–356 (2001)